Social Influence Makes Self-Interested Crowds Smarter: An Optimal Control Perspective

Yu Luo, Garud Iyengar, and Venkat Venkatasubramanian

***Abstract*— It is very common to observe crowds of individuals solving similar problems with similar information in a largely independent manner. We argue here that crowds can become "smarter," i.e., more efficient and robust, by partially following the average opinion. This observation runs counter to the widely accepted claim that the wisdom of crowds deteriorates with social influence. The key difference is that individuals are self-interested, and hence, can reject feedback that does not improve performance. We propose a control-theoretic methodology to compute the degree of social influence, i.e., the level to which one accepts the population feedback, that optimizes performance. We conducted an experiment with human subjects ($N = 194$), where the participants were first asked to solve an optimization problem independently, i.e., with no social influence. Our theoretical methodology estimates a 30% degree of social influence to be optimal, resulting in a 29% improvement in the crowd's performance. We then let the same cohort solve a new problem and have access to the average opinion. Surprisingly, we find the average degree of social influence in the cohort to be 32% with a 29% improvement in performance: In other words, the crowd self-organized into a near-optimal setting. We believe this new paradigm for making crowds "smarter" has the potential for significant impact on a diverse set of fields from population health to government planning. We include a case study to show how a crowd of states potentially could collectively learn the level of taxation and expenditure that optimizes economic growth.**

***Index Terms*— Control theory, multi-agent systems, sociotechnical systems, collective intelligence.**

I. Introduction

OFTEN, large *crowds* of decision makers are attempting to solve the same problem with similar information in a largely independent manner. For a crowd of individuals, these problems could be as simple as choosing the most appropriate product or improving personal fitness. For a crowd of local governments or nations, the problem could be optimal taxation to promote economic growth. The process of identifying the appropriate decision involves an expensive trial and error process to explore the entire space. Minimizing this search cost by coordinating and improving this *collective learning* process, by making crowds "smarter," has immense societal value.

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Optimization can be thought of as a process of balancing tradeoffs. Consider the problem of optimal taxation. Under-taxation results in insufficient funds toward public services and the functioning of the government, whereas over-taxation drives businesses to places where taxes are lower, leading once again to a deficit for the state. Local governments face similar dilemma when setting expenditure to balance between under- and over-spending. To illustrate the learning process to set the optimal taxation and expenditure, in Fig. 1, we plot the license tax as a fraction of the total state revenue from 1946 to 2014, and the secondary education expenditure as a fraction of the total state spending from 1977 to 2013. The trajectories appear to have converged in the last decade. A large majority of the states have converged to the same decision. The main question we address in this paper is whether one can accelerate convergence by making the crowd of 50 states "smarter." Even a small improvement in the convergence rate, magnified by the scale of the problem, could potentially save the nation billions of dollars while improving the overall welfare.

Using a coordinated crowd or swarm to solve complex problems is well studied in the literature. Particle swarm optimization (PSO) [3] is a widely adopted global optimization technique that uses a crowd of simple solvers to explore the fitness landscape of a problem. This swarm of PSO solvers mimics the swarming behavior observed in nature, e.g., among bees, ants, and birds. Each PSO solver revises its search direction based on its past performance and the position of the solver that observes the highest fitness. The PSO technique is very effective in solving deterministic problems that have multiple local extrema. However, PSO or any other parallel computing methodology cannot help us in improving the rate for learning in the optimal taxation and expenditure setting. The critical difference is that in the PSO setting each solver observes the same function; however, the reward or the *output* of a fitness function of an individual in a crowd is typically subjective, private, very noisy, and often, not even numerically expressible. On the other hand, the *input* to the fitness function is numerically well defined. We exploit this feature to develop a learning algorithm.

Wisdom of crowds describes the phenomenon—first introduced as *vox populi* in [4] and then rediscovered and popularized in [5]—that the average opinion of a crowd is remarkably close to the otherwise unknown truth although the opinions of individuals in the crowd are very erroneous. This phenomenon partially justifies the efficiency of polling and prediction markets, where a surveyor can gather an accurate estimate of an unknown variable by averaging over multiple independent and informed guesses. Explanations [6]–[8] for

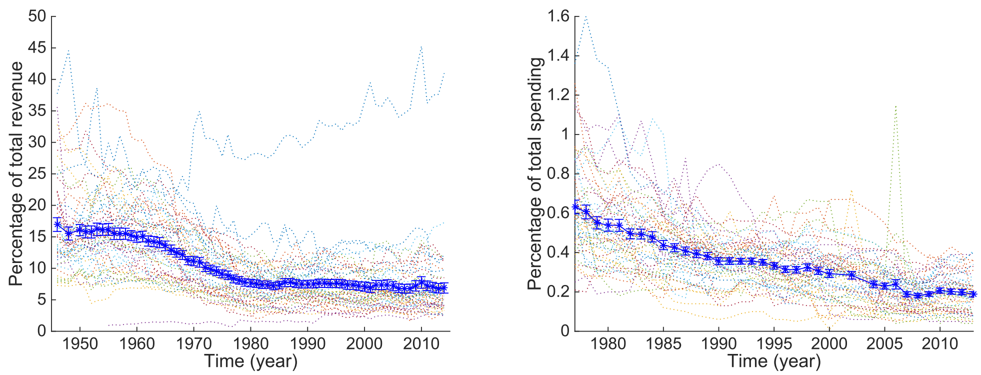

Fig. 1. State tax percentage of total revenue (total license taxes, from 1946 to 2014) [1] (left). State expenditure percentage of total spending (secondary education direct administrative expenditures, from 1977 to 2013) [2] (right). Each colored dashed line indicates the time series for one of the 50 states (and District of Columbia). The blue dotted line indicates the arithmetic mean. Error bars reflect the standard errors of the mean.

the success of the wisdom of crowds assume that individuals' estimates are unbiased and *independently* distributed [5], [9]–[13]. Social influence renders the wisdom of crowds ineffective [10], [11], [14], and in order to guarantee accuracy, interactions among the respondents should be discouraged. Since individuals make decisions solely based on their prior knowledge and expertise, some even suggest *vox expertorum*, instead of *vox populi*, to be a more suitable name [10], [15], [16].

Today, individuals are, increasingly, getting all their information from highly interconnected online social networks; thus, truly independent opinions are becoming rare. The existing literature suggests that *vox populi* should not be effective. And yet, online networks with very high degree of social interaction appear to be able to harness information effectively to benefit the individuals. We are relying on polling evermore, for selecting movies, restaurants, books, shows, etc. The polls appear to be working in identifying good options, even though the votes are highly correlated. The crowd benefits from these interactions by converging to the optimum faster. Social influence here improves, rather than undermines, the collective learning process. How does one reconcile with the previous results on the degradation of the impact of *vox populi* in the presence of social influence? *Is there an optimal degree of social influence for a learning crowd?* This is the question we address in this paper.

In this paper, we use control theory to show that *self-interested* decision makers can benefit by *partially* following the wisdom of crowds. Too little social influence prevents individuals to harness the wisdom of crowds' effect; however, too high a social influence has an adverse effect on the accuracy of the wisdom of crowds. The optimal degree of social influence balances these two effects.

We designed a human subject experiment called the "Fitness Game" that mimics the real-world situation where individuals alter their diets to improve health. By analyzing experiment results, we identify learning dynamics, determine the average degree of social influence when subjects partially follow the wisdom of crowds feedback, and calculate the optimal degree of social influence that could have maximally improved the crowd's performance.

II. Experimental Design

We conducted an online experiment on Amazon Mechanical Turk (AMT) with human subjects. There were three sets of experiments labeled set B, set N, and set S, respectively. We focus our analysis on set B ($N = 194$) only but present the final results for all three sets. Each set consists of five replications of the experiment with its unique conditions.

The participants (or players of the "Fitness Game") were asked to estimate the "diet level" that maximizes the "fitness" of a virtual character. The true relationship between the diet level and fitness was a given deterministic and concave function (i.e., there exists a unique diet level that maximizes the fitness); however, the players received a noisy value of the fitness associated with the guessed diet level. This noise, in reality, could come from other external factors such as environment and mood. The players were allowed multiple guesses, and were rewarded instantly based on the character's fitness level. The players also received monetary rewards based on their relative performances.

We conducted five replications of the "Fitness Game" for each experiment set. In replication $p \in \{1, \ldots, 5\}$, the n_p participants first entered a session where they played the game in an *open-loop* manner for 240 s (4 min). In this session, each participant entered a series of guesses (z) to best predict the unknown optimal diet level $\theta^* \in [2000, 2500]$ kcal. When a player entered a guess z for the optimal θ^*, the interface would refresh and the player would see the virtual character's fitness level (maximum is 100%) for the guessed value. The player could then enter a new value until this session ended. The term open loop indicates that individual decisions did not interact with each other; thus, the *vox populi* feedback was absent.

Subsequently, the same cohort entered the treatment session where they played the same game with a population feedback. The game was reset and a new optimal diet level θ^* was chosen. In this session, in addition to the fitness levels corresponding to their own guesses, players also received a feedback that said: "We recommend u kcal," where u denotes the average of most recent guesses from all players. This feedback was updated only when the players took actions. The players had the option of using the feedback in any manner they desired. Note that each player was unaware of other players' decisions or rewards; therefore, the system does not constitute a game, in a game-theoretic sense. If the players knew that the recommendation is the average decision, they would take the best-response action based on the aggregate state with the dynamics resulting in a mean-field game. Analyzing the game-theoretic perspective of social influence is interesting but beyond the scope of this paper (see Appendix A for detailed descriptions of the "Fitness Game" interface).

In this treatment group, we revealed the population average of the diet level to each player. Thus, the choices of the players were *not* independent. However, we allowed the players the freedom to accept, reject, or partially accept such a population feedback, i.e., set the diet level to be a combination of their individual guesses and the feedback. We call this "soft feedback" in the sense that individuals are allowed to choose the degree to which they adopt the feedback.

Luo *et al.* [17] had introduced the possibility of partial acceptance of population recommendation in the context of regulating emerging industries. In the regulatory context, we termed this as "soft" regulation in contrast to the conventional "hard" regulation where the regulated entities face fines and other punitive consequences for noncompliance. In our current setting, the individuals are allowed to partially accept the population feedback. This is in contrast to feedback in control theory, which is hard in the sense that it has to be followed. We showed that soft regulation is appropriate and efficient (and desirable) when the observed outcomes are very noisy, individual decision makers are rational utility-maximizing agents, and the agents are exploiting abundant resources, and therefore, not competing. Medical research and health optimization using large-scale social interactions, for example, via Apple's ResearchKit and CareKit [18], are examples of systems that satisfy these three conditions. The "Fitness Game" is meant to mimic these conditions.

Upon completion, participants received monetary rewards based on their relative game scores within the same cohort. We hoped to incentivize the participants in this way so that they would make rational decisions and actively optimize their virtual character's fitness, instead of making random guesses to get the participation rewards.

III. Multi-Agent Control Model

We propose the following state-space control model to describe the collective dynamics of an n-player crowd in the open-loop setting:

$$x_i(t+1) = g_i(x_i(t)) + \omega_i(t). \tag{1}$$

In the soft feedback setting, we have

$$x_i(t+1) = (1-\beta_i)(g_i(x_i(t)) + \omega_i(t)) + \beta_i u(t) \tag{2}$$

where $x_i(t)$ is the state variable of the ith player ($i = 1, \ldots, n$) at time t, $g_i(\cdot)$ is the learning function, $\omega_i(t)$ is a zero-mean random variable, $u(t)$ is the soft feedback, and β_i is the *degree* of social influence.

A. State $x_i(t)$ of the ith Player

The state variable $x_i(t) = z_i(t) - \theta^*$ is the *decision error*, i.e., difference between the individual decision $z_i(t)$ and the optimal decision θ^*. $x_i^* = 0$ indicates the optimal state (or the solution).

B. Learning Function $g_i(\cdot)$ and Noise $\omega_i(t)$

The learning function $g_i(\cdot)$ of the ith player encodes the process where the player makes a decision, observes the corresponding utility, and then updates the state. We assume that the optimal state $x_i^* = 0$ is an attracting and unique fixed point of $g_i(\cdot)$, i.e., $g_i(0) = 0$. Thus, regardless of the optimization technique or the initial decision, a player can always reach the optimum. We further assume that $g_i(x_i)$ is differentiable and $|g_i'(x_i)| < 1$, i.e., $g_i(x_i)$ is a contraction [19]. The closer the $|g_i'(x_i)|$ is to one, the slower the $g_i(x_i)$ converges. From the mean value theorem, we can also establish that $g_i(x_i)/x_i = g_i'(\delta x_i)$, where $0 \le \delta \le 1$, is strictly less than one. We define learning gain, denoted by $\tilde{g}_i' \equiv g_i(x_i)/x_i$, as the amplification of decision error.

$\omega_i(t)$ is a zero-mean random variable with variance σ_ω^2 sampled at t. It represents the impact of the error in function evaluation on the decision. Such an error can be a result of noise in measurement or external disturbance.

C. Soft Feedback $u(t)$

The players receive the population average

$$u(t) = \frac{1}{n}\sum_{j=1}^{n} x_j(t) \tag{3}$$

as feedback. Note that u denotes the decision error of the crowd. Unlike in the context of control theory, the players are not required to follow the feedback.

D. Degree of Social Influence β_i

The degree of social influence β_i denotes the weight the ith player places on the soft feedback. Setting $\beta_i = 0$ reduces the soft feedback setting in (2) to the open-loop setting in (1). We interpret β_i as the optional weight *chosen* by the ith player, as opposed to a prescription from a central planner. In the experimental setting, it is an unknown parameter that must be inferred from the observed actions of the player. In Section III-F, we compute the optimal choice for β assuming that all players are identical.

E. Convergence of the Soft Feedback Mechanism

We have previously established the following properties of soft feedback [17]. Social influence does not destabilize the system, nor does it alter the convergence provided $0 \le \beta_i < 100\%$. We can write the noiseless soft feedback dynamics by replacing $g_i(\cdot)$ with the learning gain $\tilde{g}_i'(t)$ computed at t

$$x_i(t+1) = (1-\beta_i)\tilde{g}_i'(t)x_i(t) + \beta_i u(t). \tag{4}$$

In vector form, we have

$$\mathbf{x}(t+1) = (I - B)G'(t)\mathbf{x}(t) + BS\mathbf{x}(t) \tag{5}$$

where $\mathbf{x}(t) = [x_1(t), \ldots, x_n(t)]^\top$, $B = \text{diag}(\beta_1, \ldots, \beta_n)$, $G'(t) = \text{diag}(\tilde{g}'_1(t), \ldots, \tilde{g}'_n(t))$, and $S = \frac{1}{n}\mathbf{1}\mathbf{1}^\top$. It is easy to identify that the largest eigenvalue of the matrix $(I - B)G'(t) + BS$ is *always* strictly less than one if $0 \le \beta_i < 100\%$. This implies the soft feedback dynamics also converges to the solution and is robust against bounded noise (see Appendix B for detailed proofs).

F. Efficiency of the Soft Feedback Mechanism

We define the following optimal control problem for computing the optimal degree of social influence β_i that minimizes the cost function V:

$$\min_B \; V(B, T) = \mathbb{E}\left[\sum_{t=0}^{T-1} \frac{1}{n}\mathbf{x}(t)^\top \mathbf{x}(t)\right]$$
$$\text{s.t. } \mathbf{x} := (I - B)(G'\mathbf{x} + \boldsymbol{\omega}) + BS\mathbf{x} \tag{6}$$

where we drop indices t and $t+1$ here because of the space constraint (instead we use := to indicate the recursive process of updating $\mathbf{x}$) and $\boldsymbol{\omega} \equiv \boldsymbol{\omega}(t) = [\omega_1(t), \ldots, \omega_n(t)]^\top$ denotes a series of noise vectors. The above optimal control problem minimizes the cumulative expected mean squared error (MSE) over a finite time horizon T.

The time evolution of MSE in (6) depends on two factors: the rate of convergence controlled by $((I-B)G'(t)+BS)\mathbf{x}(t)$ and the noise reduction controlled by $(I-B)\boldsymbol{\omega}(t)$. A stronger social influence, i.e., high β_i, leads to less noise. The contraction effect depends on the largest singular value of the matrix $(I-B)G'(t)+BS$. Given $G'(t)$ and S, there always exists a social influence profile B such that the largest singular value is minimized.

The overall problem is a nonconvex optimization and difficult to solve analytically. In addition, both $\tilde{g}'_i$ and β_i are heterogeneous among players. In order to understand the fundamental tradeoffs in this problem, we assume that the aggregate dynamics of the system can be described by n identical "representative agents" with learning function $g_i(x_i) \equiv g(x) = \tilde{g}x$ and degree of social influence $\beta_i \equiv \beta$. With this assumption, we can reduce the original dynamics into the following linear stochastic dynamics:

$$\mathbf{x}(t+1) = [(1-\beta)\tilde{g} + \beta S]\mathbf{x}(t) + (1-\beta)\boldsymbol{\omega}(t). \tag{7}$$

In Appendix B, we show that the expected MSE satisfies

$$\mathbb{E}[\text{MSE}(t+1)] \le m^2\mathbb{E}[\text{MSE}(t)] + (1-\beta)^2\sigma_\omega^2 \tag{8}$$

where $m = (1-\beta)|\tilde{g}| + \beta$. Define

$$\overline{M}(0) \equiv \mathbb{E}[\text{MSE}(0)],$$
$$\overline{M}(t+1) \equiv m^2\overline{M}(t) + (1-\beta)^2\sigma_\omega^2. \tag{9}$$

Then

$$\overline{V}(\beta, T) \equiv \mathbb{E}\left[\sum_{t=0}^{T-1} \overline{M}(t)\right] \tag{10}$$

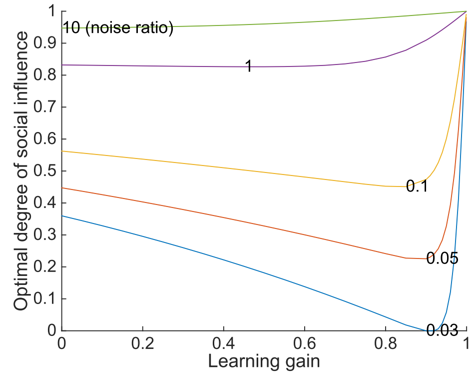

Fig. 2. Optimal degree of social influence (minimax) as a function of the learning gain when the noise-to-initial-MSE ratio is 0.03, 0.05, 0.1, 1, or 10. The general trend is that a moderately strong social influence is desirable if the system is uncertain (high noise-to-initial-MSE ratio) or the learning gain is low (fast open-loop convergence). An interesting observation is that as the learning gain crosses a certain threshold (e.g., 0.9), the optimal degree of social influence rapidly increases as the learning gain increases. For a high learning gain, the contraction becomes insensitive to the change in β while the noise reduction still does.

is a convex upper bound for the cost $V(\beta, T)$. From (9), it follows that:

$$\frac{\overline{V}(\beta, T)}{\text{MSE}(0)} = \frac{1-m^{2T}}{1-m^2} + T\frac{(1-\beta)^2}{1-m^2}\frac{\sigma_\omega^2}{\text{MSE}(0)} - \frac{1-m^{2T}}{1-m^2}\frac{(1-\beta)^2}{1-m^2}\frac{\sigma_\omega^2}{\text{MSE}(0)}. \tag{11}$$

We propose computing an approximately optimal control by *minimizing* the *maximum* cost $\overline{V}(\beta, T)$. Denote the conservative minimax solution

$$\beta^*_{\text{MM}} = \arg\min_\beta \overline{V}(\beta, T). \tag{12}$$

In Fig. 2, we plot the value of β^*_{MM} as a function of the noise-to-initial-MSE ratio $\sigma_\omega^2/\text{MSE}(0)$ and the characteristic learning gain $\tilde{g}$ (given $T = 30$). A moderate social influence is optimal when systems are uncertain and one needs the system to equilibrate quickly.

IV. Results

A. Wisdom of Crowds' Effect

We begin with the analysis of the wisdom of crowds' effect. We plot the time series of each individual player's decision error (x_i) as well as that of the wisdom of crowds (u) in Fig. 3 (similar to the state tax and expenditure time series in Fig. 1). The performance of the wisdom of crowds is clearly superior: u steadily and quickly reaches the solution within the first minute while individual players lag behind.

Fig. 3 also confirms the behavior observed in the literature: The wisdom of crowds significantly outperforms the individual estimates, but such an effect is weakened by social influence. The average in the soft feedback setting [the red curve in Fig. 3 (right)] slightly lags behind that in the open loop [the blue curve in Fig. 3 (left)].

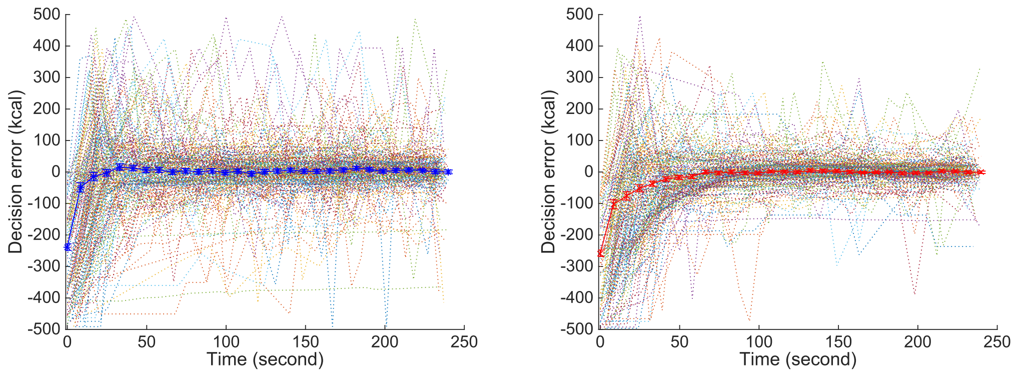

Fig. 3. Learning process of each individual player (time series of x_i) and the wisdom of crowds (time series of u). Open-loop setting (left). Soft feedback setting (right). Each colored dashed line represents an individual participant's time series of decision error. The solid line is the arithmetic average of individual decision errors (i.e., wisdom of crowds). Error bars reflect the standard errors of the mean.

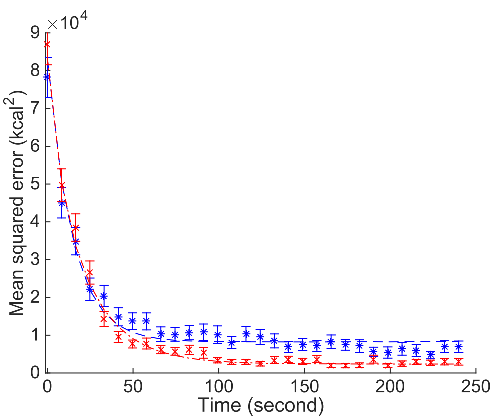

Fig. 4. MSE progression. Blue asterisks (respectively, red crosses) are the MSE values sampled at different points in time ($T = 30$) in the open-loop (respectively, soft feedback) setting. The dashed lines are simulation results based on models from system identification (Section IV-C).

B. Improvement from Soft Feedback

Next, we analyze how soft feedback improves the crowd's learning performance. By visually inspecting Fig. 3, we observe the narrowing of individual error distribution in the soft feedback setting: There are fewer extreme errors than those in the open-loop setting; most guesses are confined within ±100 kcal around optimum. In contrast, there are a significant number of players making completely off guesses (±500 kcal) in the open loop (even toward the end of sessions).

In Fig. 4, we plot the MSE time series to quantitatively assess the crowd's performance. The total MSE is approximately 30% lower in the soft feedback setting than in the open-loop setting. Unlike the deterioration in the performance of wisdom of crowds, here social influence improves convergence and reduces the effect of noise. *The critical feature of soft feedback is that the players can ignore the feedback.* Since self-interested individuals reject feedbacks that appear unhelpful, the self-filtered social feedback significantly improves performance.

The observed improvement from soft feedback indicates that, without external interference, partially following the average opinion helped the players solve the "Fitness Game" problems. In the next section, we will characterize the system, estimate the extent social influence present in the experiment, and compute the optimal degree of social influence that would have optimized the crowd's performance.

TABLE I

OPTIMAL DEGREE OF SOCIAL INFLUENCE

Type	Value	ΔMSE	Remark
$\hat{\beta}$	32%	29%	Observed
β^*_{MM}	23%	27%	Minimax
β^*_{MC}	30%	29%	MC, true optimum
$\hat{\beta}(d)$	$e^{-0.011d}$	30%	Observed (profile)
$\beta^*_{\mathrm{MC}}(d)$	$e^{-0.026d}$	47%	MC, true optimum (profile)
$\beta^*_{\mathrm{MM}}(t)$	-	39%	Minimax (dynamic)

C. System Identification

We assumed the learning function $g_i(x_i) \equiv g(x) = \tilde{g}x$ and the degree of social influence $\beta_i \equiv \beta$. The estimates $\hat{g}(x) = 0.75x$ and $\hat{\sigma}_\omega = 60$ ($r^2 = 0.97$) were computed using the open-loop results. Suppose $\tilde{g}$ and σ_ω are known. Then one can simulate the open-loop dynamics described in (1) by drawing random variable $\omega_i(t)$ from a normal distribution with standard deviation σ_ω. We approximated the time evolution of the expected MSE for a given $(\tilde{g}, \sigma_\omega)$ using 5000 Monte Carlo (MC) samples. The estimate $(\hat{\tilde{g}}, \hat{\sigma}_\omega)$ was set equal to the value for $(\tilde{g}, \sigma_\omega)$ that minimized the squared difference between simulated MSE values and measured MSE data. The corresponding MSE evolution is plotted in Fig. 4.

The estimate $\hat{\beta} = 32\%$ ($r^2 = 0.99$) for the degree of social influence was computed using the results where the players received the population feedback. The corresponding MSE evolution is plotted in Fig. 4. Following the studies [20], [21] that have established that people rely more on themselves when the opinions of others are very dissimilar, we computed an "opinion distance" function $\beta(d)$, where $d = |g(x) - u|$ is the distance of an individual decision from the population feedback. We found it to be $\hat{\beta}(d) = \exp(-0.011d)$ ($r^2 = 0.98$).

D. Optimal Degree of Social Influence

Given the estimates $\hat{g}(x)$ and $\hat{\sigma}_\omega$, one can compute the optimal degree of social influence β^* that, hypothetically, would optimize the soft feedback performance. The results

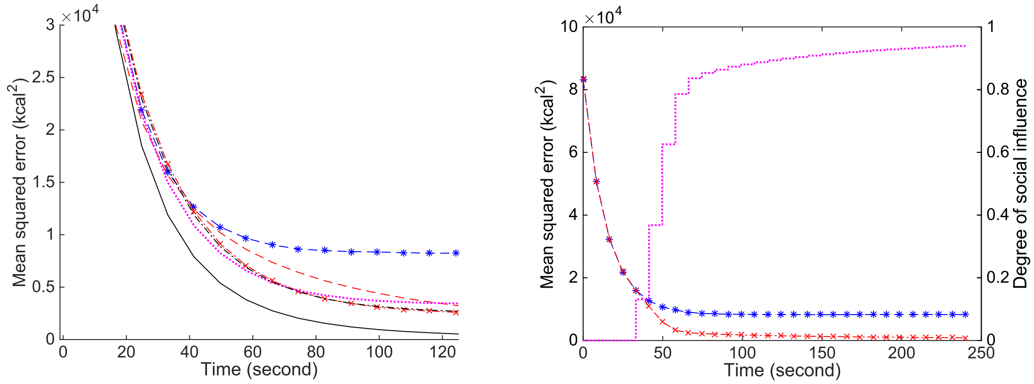

Fig. 5. MC simulation of the expected MSE time series. The blue (respectively, red) dashed line with asterisks (respectively, crosses) is the simulation of the open-loop (respectively, soft feedback) MSE. The red dashed line is the simulation of the soft feedback MSE with social influence profile estimate $\hat{\beta}(d)$; the magenta dotted line is the simulation of MSE with optimal degree of social influence β^*_{MM} through minimax; the black dashed-dotted line is with true optimal degree of social influence β^*_{MC}; and the black solid line is with true optimal social influence profile $\beta^*_{\mathrm{MC}}(d)$ (left). The red dashed line with crosses is with dynamic social influence $\beta^*_{\mathrm{MM}}(t)$ and the magenta dotted line is the dynamic social influence time series (right).

are summarized in Table I, and the associated MSE time series is displayed in Fig. 5. We first consider the case where the degree of social influence β is fixed. The empirical estimate $\hat{\beta}$ of social influence *observed* from data is listed as a reference. The *minimax* social influence β^*_{MM} was calculated by minimizing the RHS in (11), i.e., optimizing the worst case cumulative expected MSE. The *MC* estimate β^*_{MC} was calculated by minimizing the total MSE in (6) with the expectation approximated by an MC estimate. We regard β^*_{MC} as the true optimal degree of social influence. In Table I, the column labeled ΔMSE lists the decrease of the cumulative expected MSE from the open loop to the soft feedback setting. The performances of the empirical estimate $\hat{\beta}$, the minimax estimate β^*_{MM}, and the optimal value β^*_{MC} are quite close. It is comforting to know that the social influence present in the experiment was close to the optimum.

We expect the degree of social influence—a function of the opinion distance (β *profile*) or a function of time (*dynamic* β)—to likely improve convergence. The $\hat{\beta}(d)$ profile estimated from data results in $\Delta\mathrm{MSE} = 30\%$, which is not distinguishable from the performance of a constant β. However, the optimal β profile $\beta^*_{\mathrm{MC}}(d)$ with $\Delta\mathrm{MSE} = 47\%$ is significantly superior. The performance of the optimal dynamic minimax social influence $\beta^*_{\mathrm{MM}}(t)$ is also listed in Table I. Since we do not have evidence to suggest the subjects used a dynamic value for β, and the performance of $\hat{\beta}(d)$ is close to $\hat{\beta}$, we assume that the subjects used the constant $\hat{\beta}$ for the rest of our results.

E. U.S. State Tax and Expenditure Case Study

Next, we apply this control-theoretic analysis to the state tax and expenditure case study. There are 50 states in the United States ($n = 50$, or $n = 51$ if we consider the District of Columbia). Each state here is an intelligent decision maker and constantly revises its tax and expenditure policies. The goal is to maximize the overall wellbeing (economic growth, political stability, etc.). In this case study, the policy or strategy $z_i(t)$ for the ith state here is the percentage of a particular tax revenue (respectively, expenditure) from the total revenue (respectively, spending). Such percentage reflects the relative importance of a particular tax item (respectively, expenditure item). The main question is whether one can accelerate the convergence toward some optimal taxation or expenditure thereby making the 50 states collectively "smarter."

We first make a few simplifying assumptions of the problem. From Fig. 1, we observe that the *average* levels of taxation or expenditure among all the states have reached steady-state values in the last decade. We assume based on [4] and [5] that average is a good estimate of the true optimal policy. The MSE, for example, can be defined as the mean squared differences between individual policy values and the equilibrium average values. The optimization problem is therefore finding the optimal degree of social influence such that the cumulative MSE is minimized over the time periods of 1946–2014 (tax) and 1977–2013 (expenditure). We identify the systems (e.g., representative learning gains and noise-to-initial-MSE ratios) using the same method described in Section IV-C.

The results are displayed in Table II. The learning gains of the states are all very close to one, i.e., in a noiseless setting, the convergence is very slow. A possible explanation is that drastic change of tax and expenditure strategies is either prohibited or discouraged. A larger noise (see T09 and E065) or a smaller learning gain (see T20 and E065) calls for a larger optimal degree of social influence, which is consistent with the results presented in Fig. 2. The improvement from soft feedback ranges from 14% to 73%. As mentioned earlier, even a small improvement could make a significant difference in the nation's overall welfare. Note that this thought experiment with its simplifying assumptions is not intended to simulate the actual decision-making processes of local policymakers; but rather, we are interested in applying our model to real-world situations and studying how the learning gain and the noise ratio affect the efficiency of soft feedback.

V. DISCUSSION

There is a fundamental difference between *vox populi* and the soft feedback mechanism proposed in this paper.

TABLE II

ALL RESULTS

Description	Duration	Crowd size (n)	Horizon (T)	Learning gain ($\hat{g}$)	Noise ($\hat{\sigma}_\omega$)	r^2	Noise ratio	Optimal β	ΔMSE
The "Fitness Game" (Set B)	0—240s	39	30	0.75	60	0.97	5%	30%	29%
The "Fitness Game" (Set N)	0—240s	41	30	0.7	57	0.98	4%	32%	25%
The "Fitness Game" (Set S)	0—240s	9	30	0.65	51	0.98	3%	30%	17%
Total Gen Sales Tax (T09)	1946—2014	50	69	0.96	4	0.89	3%	35%	73%
Total License Taxes (C118)	1946—2014	50	69	0.97	0.82	0.89	0.4%	14%	34%
Alcoholic Beverage Lic (T20)	1946—2014	50	69	0.93	0.04	0.99	0.09%	20%	34%
Individual Income Tax (T40)	1946—2014	50	69	0.98	2.9	0.86	1%	14%	32%
Educ-NEC-Dir Expend (E037)	1977—2013	51	37	0.96	0.097	0.85	1%	28%	54%
Emp Sec Adm-Direct Exp (E040)	1977—2013	51	37	0.93	0.037	0.99	0.6%	11%	14%
Total Highways-Dir Exp (E065)	1977—2013	51	37	0.93	0.76	0.89	3%	31%	53%
Liquor Stores-Tot Exp (E107)	1977—2013	51	37	0.95	0.17	0.95	1%	42%	67%

Even though both come under the umbrella of "collective intelligence," the *vox populi* aggregates the wisdom of *experts* while the latter harnesses the wisdom of *learners*. Experts base their opinions on prior knowledge. Such knowledge comes from experience and beliefs, which are unlikely to change. Independency and diversity of opinions prevent the "groupthink" behavior—undesirable convergence of individual estimates [22]. In this setting, social influence, which violates independency, reduces the accuracy of the wisdom of crowds.

Learners, on the other hand, *revise* their decisions by interacting with the problem as well as other learners. Consider, for example, flocking birds. The birds have to adapt to changing weather; they gather local information, follow their closest neighbors, and revise directions constantly [23]. In this collective learning environment, individuals, like the flocking birds, are *both* respondents who generate new information and surveyors who poll their social networks to improve decisions.

It appears that a degree of social influence of 30% is robust across many different scenarios. In Table II, the optimal degree of social influence ranges from 30% to 32% for the "Fitness Game" experiment. Prior literature [20], [24]–[27] also reports 30% to be the commonly observed degree of social influence on average. Whether this value is a mere coincidence requires further investigation.

The self-interested filtering of the feedback is key to ensuring the accuracy and efficiency of the soft feedback mechanism. Individuals will reject the feedbacks that appear useless. The experimentally observed magnitude of soft feedback is close to the theoretically predicted value for the optimal degree of social influence. This discovery suggests the promise of soft feedback for challenging real-world problems that require collective learning and action.

APPENDIX A
"FITNESS GAME"

All experiments have been approved by the Institutional Review Board of Columbia University (Protocol Number: IRB-AAAQ2603). We developed the "Fitness Game" using Google Apps Script and conducted the experiments on AMT. All the data were stored in Google Sheets. Once the players accepted the task on AMT, they were first asked to carefully read the game instructions (see Fig. 6). The total task duration was 10 min. The open-loop (game level 1) and soft feedback (game level 2) sessions lasted precisely 4 min each. Players who wished to practice could enter the practice mode (game level 0) any time before open-loop session began.

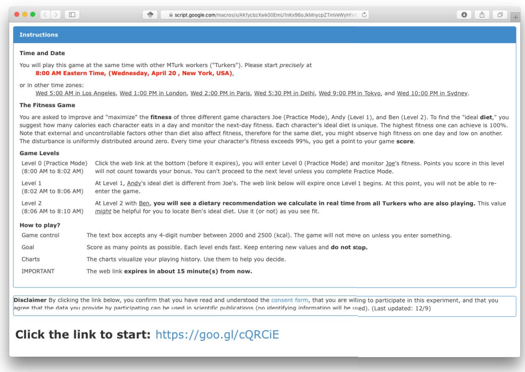

Fig. 6. Instructions.

After completing both open-loop and soft feedback sessions, the players received a message about compensation information.

The interactive app (see Fig. 7) consists of the following components. The top-left panel shows the number of attempted guesses, the most recent guess, the fitness level, and the latest score. The panel changes from red to green whenever the player earns one point. In the soft feedback session, an additional message recommends the current *vox populi* population feedback [see Fig. 7 (bottom)]. The top-right panel records latest game scores. The bottom-left scatter chart plots the 10 most recent entries (fitness versus diet). The bottom-right line chart plots the fitness history of the 10 most recent entries.

The virtual character's random fitness level $f(x)$ as a function of the decision error $x = z - \theta^*$ was given by

$$f(x) = f_0 - \left(\frac{x}{\kappa}\right)^2 + \nu$$

where $f_0 = 98\%$ is the expected maximum fitness the virtual character can achieve, $\kappa = 500$ kcal is the scale parameter of the fitness function, and ν is a sample from a random variable uniformly distributed over $[-2\%, 2\%]$. The player was awarded one score point whenever the guess led to a fitness level of 99% or higher.

APPENDIX B
MATHEMATICAL MODEL AND PROOFS

We first begin with the noiseless dynamics and then extend the model to include noise. The noiseless dynamics for the

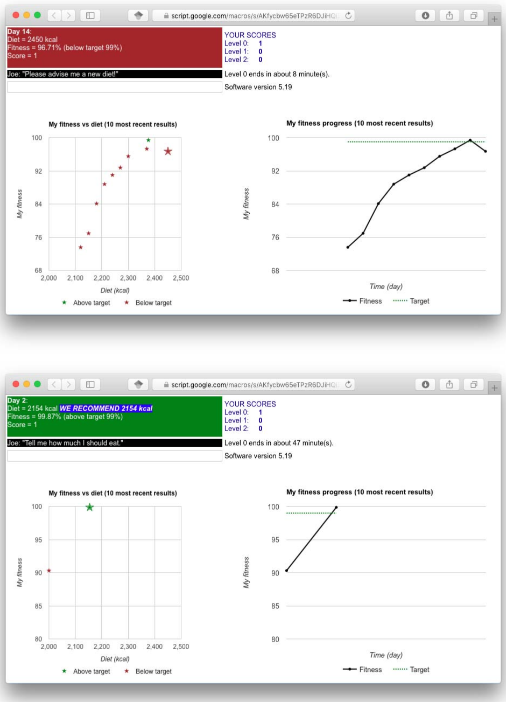

Fig. 7. Game interface.

n-player "Fitness Game" is as follows:

$$x_i(t+1) = (1-\beta_i)g_i(x_i(t)) + \beta_i u(t)$$

where $x_i(t)$ is the ith player's state (decision error), i.e., deviation from optimum θ^* at time t, and restricted to belong to a bounded set $\mathbb{X} \subseteq \mathbb{R}$, the learning function $g_i(\cdot)$ denotes the player's own state update process, and $\beta_i \in [0,1]$ (degree of social influence) is the weight player i puts on the soft feedback $u(t) = \frac{1}{n}\sum_{j=1}^{n} x_j(t)$. Note that in this paper, we refer to β as percentage. The individual learning functions $\{g_i(x_i) : 1 \le i \le n\}$ are assumed to satisfy the following regularity condition.

Assumption 1: For all $i \in \{1,\ldots,n\}$, the function $g_i(\cdot)$ is differentiable, $x_i^* = 0$ is the unique attracting fixed point of $g_i(\cdot)$, and furthermore, $g_i(\cdot)$ is a contraction, i.e., $|g_i'(x_i)| < 1$ for all $1 \le i \le n$ and $x_i \in \mathbb{X}$.

This assumption is motivated by the fact that all players converged to the optimal point in the open-loop setting independent of the starting guess.

Let $\mathbf{x} \equiv [x_1,\ldots,x_n]^\top \in \mathbb{X}^n$ denote the state vector for the n players. The soft feedback map for the vector $\mathbf{x}$ is given by $\mathbf{x}(t+1) = \mathbf{h}(\mathbf{x}(t))$

$$\mathbf{h}(\mathbf{x}) = \mathrm{diag}((1-\beta_1)g_1(x_1),\ldots,(1-\beta_n)g_n(x_n)) + \frac{1}{n}\begin{bmatrix}\beta_1\\ \vdots\\ \beta_n\end{bmatrix}\mathbf{1}^\top\mathbf{x}.$$

We first show that the state vector $\mathbf{x}(t)$ converges to $\mathbf{x}^* = \mathbf{0}$ if the functions $\{g_i(x_i) : 1 \le i \le n\}$ satisfy Assumption 1 and $0 \le \max_i \beta_i < 1$.

Theorem 1: The spectral radius $\rho(J(\mathbf{x}))$ of the Jacobian matrix of the soft feedback map $\mathbf{h}(\mathbf{x})$ satisfies $\rho(J(\mathbf{x})) \le m = \max_{1\le i\le n, x_i\in\mathbb{X}}\{(1-\beta_i)|g_i'(x_i)| + \beta_i\} < 1$.

Proof: The Jacobian of $\mathbf{h}(\mathbf{x})$ is

$$J(\mathbf{x}) = \mathrm{diag}\left((1-\beta_1)g_1'(x_1),\ldots,(1-\beta_n)g_n'(x_n)\right) + \frac{1}{n}\begin{bmatrix}\beta_1\\ \vdots\\ \beta_n\end{bmatrix}\mathbf{1}^\top.$$

The induced ∞-norm $\|J(\mathbf{x})\|_\infty$ of the Jacobian J satisfies

$$\begin{aligned}\|J(\mathbf{x})\|_\infty &= \max_{\|\mathbf{v}\|_\infty=1}\|J(\mathbf{x})\mathbf{v}\|_\infty\\ &= \max_{\|\mathbf{v}\|_\infty=1}\max_{1\le i\le n}|J_i(\mathbf{x})\mathbf{v}|\\ &= \max_{\|\mathbf{v}\|_\infty=1}\max_{1\le i\le n}\left[(1-\beta_i)\left|g_i'(x_i)\right|v_i + \frac{1}{n}\beta_i(\mathbf{1}^\top\mathbf{v})\right]\\ &\le m(\mathbf{x})\end{aligned}$$

where $J_i(\mathbf{x})$ denotes the ith row of the Jacobian $J(\mathbf{x})$. The result follows from noting that $\rho(J(\mathbf{x})) \le \|J(\mathbf{x})\|_\infty \le m(\mathbf{x})$. It is easy to see that $m(\mathbf{x}) < 1$ whenever $\max_i \beta_i < 1$. □

This result immediately implies that $\mathbf{x}^* = 0$ is an asymptotically stable fixed point of the map $\mathbf{h}(\mathbf{x})$.

Theorem 2: The fixed point $\mathbf{x}^* = \mathbf{0}$ of the map $\mathbf{h}$ is robust when subjected to bounded disturbances.

Proof: Let $V(\mathbf{x}) = \|\mathbf{x}\|_\infty$. Since $\mathbf{h}(\mathbf{0}) = \mathbf{0}$, the mean value theorem implies that

$$\mathbf{h}(\mathbf{x}) = \begin{bmatrix}J_1(\delta_1\mathbf{x})\\ \vdots\\ J_n(\delta_n\mathbf{x})\end{bmatrix}\mathbf{x}$$

for some $\delta_i \in [0,1]$, $i = 1,\ldots,n$, and $J_i(\delta_i\mathbf{x})$ denotes the ith row of the Jacobian of $\mathbf{h}(\delta_i\mathbf{x})$. Thus

$$\begin{aligned}V(\mathbf{h}(\mathbf{x})) &= \|\mathbf{h}(\mathbf{x})\|_\infty\\ &= \max_{1\le i\le n}|J_i(\delta_i\mathbf{x})\mathbf{x}|\\ &\le \left(\max_{1\le i\le n}\|J(\delta_i\mathbf{x})\|_\infty\right)\|\mathbf{x}\|_\infty\\ &\le \left(\max_{1\le i\le n} m(\delta_i\mathbf{x})\right)\|\mathbf{x}\|_\infty\\ &< \|\mathbf{x}\|_\infty\end{aligned}$$

where the first inequality follows from the definition of $\|J(\delta_i\mathbf{x})\|_\infty$.

Since the continuous function $V(\mathbf{x})$ is a Lyapunov function for $\mathbf{h}$, the result follows from standard results in stability theory [28]. □

In the rest of this section, we will assume that β_i values are identically equal to β.

Theorem 3: Suppose β_i values are all identically equal to β. Then $\|\mathbf{h}(\mathbf{x})\|_2 \le m\|\mathbf{x}\|_2$, where $m = (1-\beta)\max_{1\le i\le n, x_i\in\mathbb{X}}|g_i'(x_i)| + \beta$.

Proof: Using the mean value theorem, one can write

$$\mathbf{h}(\mathbf{x}) = \left((1-\beta)\,\mathrm{diag}\left(g_1'(\delta_1 x_1),\ldots,g_n'(\delta_n x_n)\right) + \frac{\beta}{n}\mathbf{1}\mathbf{1}^\top\right)\mathbf{x}$$

where $\delta_i \in [0,1]$ for $i = 1,\ldots,n$. Let $G' = \mathrm{diag}(g_1'(\delta_1 x_1),\ldots,g_n'(\delta_n x_n))$ and $J = (1-\beta)G' + \frac{\beta}{n}\mathbf{1}\mathbf{1}^\top$. Then

$$\begin{aligned}\|G'\|_2^2 &= \max_{\|\mathbf{v}\|_2=1} \|G'\mathbf{v}\|_2^2\\ &= \max_{\|\mathbf{v}\|_2=1} \sum_{i=1}^n \left|g_i'(\delta_i x_i)\right|^2 v_i^2\\ &\le \max_{1\le i\le n} \left|g_i'(\delta_i x_i)\right|^2\\ &\le \max_{1\le i\le n, x_i\in\mathbb{X}} \left|g_i'(x_i)\right|^2.\end{aligned}$$

Thus, $\|G'\|_2 \le \max_{1\le i\le n, x_i\in\mathbb{X}} |g_i'(x_i)|$. Therefore

$$\begin{aligned}\|J\|_2^2 &= \max_{\|\mathbf{v}\|_2=1} \|J\mathbf{v}\|_2^2\\ &= \max_{\|\mathbf{v}\|_2=1} \Bigg\{(1-\beta)^2\|G'\mathbf{v}\|_2^2 + \frac{\beta^2}{n^2}(\mathbf{1}^\top\mathbf{v})^2\|\mathbf{1}\|_2^2\\ &\qquad + \frac{2\beta(1-\beta)}{n}(\mathbf{1}^\top\mathbf{v})(\mathbf{1}^\top G'\mathbf{v})\Bigg\}\\ &\le (1-\beta)^2\|G'\|_2^2 + \beta^2\\ &\quad + \frac{2\beta(1-\beta)}{n}\max_{\|\mathbf{v}\|_2=1}|\mathbf{1}^\top\mathbf{v}|\max_{\|\mathbf{v}\|_2=1}|\mathbf{1}^\top G'\mathbf{v}|\\ &\le (1-\beta)^2\|G'\|_2^2 + \beta^2\\ &\quad + \frac{2\beta(1-\beta)}{\sqrt{n}}\|\mathbf{1}\|_2\max_{\|\mathbf{v}\|_2=1}\|G'\mathbf{v}\|_2\\ &= (1-\beta)^2\|G'\|_2^2 + \beta^2 + 2\beta(1-\beta)\|G'\|_2\\ &= m^2.\end{aligned}$$

Since $\mathbf{h}(\mathbf{x}) = J\mathbf{x}$, it follows that $\|\mathbf{h}(\mathbf{x})\|_2 = \|J\mathbf{x}\|_2 \le \|J\|_2\|\mathbf{x}\|_2 \le m\|\mathbf{x}\|_2$. □

Next, we introduce noise in the game dynamics. Let $\{\boldsymbol{\omega}(t) \in \mathbb{R}^n : t \ge 0\}$ denote an independent and identically distributed (IID) sequence of random vectors where $\boldsymbol{\omega}(t) = [\omega_1(t),\ldots,\omega_n(t)]^\top$, and each $\omega_i(t)$ is an IID sample of a zero-mean random variable with variance σ_ω^2. The noisy game dynamics is given by

$$x_i(t+1) = (1-\beta_i)(g_i(x_i(t)) + \omega_i(t)) + \beta_i u(t)$$

that is, we replace $g_i(x_i(t))$ by the noisy state update $g_i(x_i(t)) + \omega_i(t)$. This modification models the fact that the players sample a noisy version of the fitness function, and use these noisy samples to generate the update; therefore, we expect the state update to be noisy. Note that the noise is *not* measurement noise, rather noise in the function evaluation.

Define the MSE

$$\mathrm{MSE}(t) = \frac{1}{n}\sum_{i=1}^n x_i(t)^2 = \frac{1}{n}\|\mathbf{x}(t)\|_2^2.$$

Then

$$\begin{aligned}&\mathbb{E}[\mathrm{MSE}(t+1) \mid \mathbf{x}(t)]\\ &\quad= \frac{1}{n}\mathbb{E}\big[\|\mathbf{x}(t+1)\|_2^2 \mid \mathbf{x}(t)\big]\\ &\quad= \frac{1}{n}\mathbb{E}\big[\|\mathbf{h}(\mathbf{x}(t)) + (1-\beta)\boldsymbol{\omega}(t)\|_2^2 \mid \mathbf{x}(t)\big]\\ &\quad= \frac{1}{n}\|\mathbf{h}(\mathbf{x}(t))\|_2^2 + \frac{(1-\beta)^2}{n}\mathbb{E}\big[\|\boldsymbol{\omega}(t)\|_2^2\big] \qquad (13)\\ &\quad\le \frac{m^2}{n}\|\mathbf{x}(t)\|_2^2 + (1-\beta)^2\sigma_\omega^2 \qquad (14)\\ &\quad= m^2\,\mathrm{MSE}(t) + (1-\beta)^2\sigma_\omega^2 \qquad (15)\end{aligned}$$

where (13) follows from the fact that $\boldsymbol{\omega}(t)$ is independent of $\mathbf{x}(t)$, and (14) follows from the bound in Theorem 3. Iterating the bound (15), we get

$$\mathbb{E}[\mathrm{MSE}(t)] \le m^{2t}\mathrm{MSE}(0) + \frac{(1-\beta)^2(1-m^{2t})}{(1-m^2)}\sigma_\omega^2.$$

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