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How much inequality in income is fair? A microeconomic game theoretic perspective

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HIGHLIGHTS

- Fairest inequality of income is a lognormal distribution under ideal conditions.
- Ideal free market can "discover" the fairest distribution in practice, if allowed.
- Scandinavia is close to ideal inequality for the bottom 99% of the population.
- US was closer to ideal inequality in 1945–75 than it is now for the bottom 90%.
- Deep connection between potential game theory and statistical mechanics via entropy.

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ABSTRACT

The increasing inequality in income and wealth in recent years, and the associated excessive pay packages of CEOs in the US and elsewhere, is of growing concern among policy makers as well as the common person. However, there seems to be no satisfactory answer, in conventional economic theories and models, to the fundamental questions of what kind of income distribution we ought to see, at least under ideal conditions, in a free market environment, and whether this distribution is fair. We propose a novel microeconomic game theoretic framework that addresses these questions and proves that the lognormal distribution is the fairest inequality of pay in an organization comprising of homogeneous agents, under ideal free market conditions at equilibrium. We also show that for a population of two different classes of agents, the equilibrium distribution is a combination of two different lognormal distributions where one of them, corresponding to the top ∼3–5% of the population, can be misidentified as a Pareto distribution. We compare our predictions with empirical data on global income inequality trends provided by Piketty and others. Our analysis suggests that the Scandinavian countries, and to a lesser extent Switzerland, Netherlands and Australia, have managed to get close to the ideal distribution for the bottom ∼99% of the population, while the US and UK remain less fair at the other extreme. Other European countries such as France and Germany, and Japan and Canada, are in the middle. Our theory also shows the deep and direct connection between potential game theory and statistical mechanics through entropy, which we identify as a measure of fairness in a distribution. This leads us to propose the *fair market hypothesis*, that the self-organizing dynamics of the ideal free market, i.e., Adam Smith's "invisible hand", not only promotes efficiency but also maximizes fairness under the given constraints.

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1. Introduction

In recent years, there has been growing concern over the widening inequality in income and wealth distributions in the US and elsewhere [1–5]. The statistics are troubling — for instance, as of 2012, the top 1% of households in the US owned 41.8% of all privately held wealth [6], and it had risen from a low of about 20% in 1976 [7].

An important source of the wealth inequality is a similar trend in the income and pay (or wage) distributions. Income remains highly concentrated, with the top 1% of income earners received 17.9% of all income in 2012 in the US, and that is up from 12.8% in 1982 [7,8]. A related trend of equally great concern is the excessive pay packages for CEOs which are reflected in the extraordinarily high CEO pay ratios, particularly in the US [9,10]. There is much discussion both in academic literature and popular press about what all these mean, what the consequences are, and what can or should be done about it [1–5,11–16].

Obviously, before policy actions, if any, are taken to address these challenges, we need to understand more deeply why and how such inequalities occur. Since different people have *different* abilities and therefore make *different* contributions in a society, naturally, we do expect people to be compensated *unequally*, commensurate with their contributions. Hence, we would expect to see unequal distributions in income and in wealth. So, a certain *level of inequality* is to be expected. But, at the risk of sounding oxymoronic, *what is the fairest level of inequality*? In particular, in an *ideal free market environment*, *what is the level of inequality we ought to see*? This is the question we address in this paper.

While there is extensive empirical literature on income and wealth distributions, and we cite only a sample here [3,6,9,11,12,17–20], there is no satisfactory answer to this question in conventional economic theories and models. Empirical observations are obviously very important, but it would be quite helpful to complement them with a theoretical framework that provides a new useful perspective and analytical insight. From a theoretical perspective, two fundamental questions one would like answered are: *What kind of pay distribution should arise, under ideal conditions, in a free market environment comprising of utility maximizing employees and profit maximizing companies? Is this distribution fair?*

The answers to these questions can serve as a fundamental benchmark against which we can evaluate the distributions seen in real life. In the absence of such a reference framework, the conclusions we reach by relying on empirical observations alone are likely to be incomplete, in an important manner. This benchmark can help us measure and better understand the deviations caused by nonidealities in the real world, and to develop appropriate policy frameworks and incentive structures to try to correct the inequalities. It can give us a quantitative basis for understanding and developing rational tax policies, pay packages for executives, and so on. Our objective in this paper is to develop such a benchmark by proposing a novel microeconomic framework that predicts and explains the emergence of an appropriate pay distribution under ideal free market conditions.

When one explores outside mainstream economics in search of answers to these questions, one finds that there has been much work, in the past decade or so, in the econophysics community to model income and wealth distributions by applying concepts and techniques from statistical mechanics [20–35]. While these models are quite interesting and instructive, they have not, however, bridged the rather wide conceptual gulf that exists between economics and econophysics [36,37], particularly in two crucial areas. One, the typical particle model of agent behavior in econophysics assumes agents to have nearly "zero intelligence", acting at random, with no intent or purpose. This does not sit well with an extensive body of economic literature spanning several decades, where one models, in the ideal case, a perfectly rational agent whose goal is to maximize its utility or profit by acting strategically, not randomly. From the perspective of an economist, it is quite reasonable to ask "How can theories and models based on the collective behavior of purpose-free, random, molecules explain the collective behavior of goal-driven, optimizing, strategizing men and women?"

Another conceptual stumbling block is the role of entropy in economics. In statistical thermodynamics, equilibrium is reached when entropy, which is a measure of randomness or uncertainty, is maximized. So, an economist wonders, why would maximizing randomness or uncertainty be helpful in economic systems? We all know that markets are stable, and function well, when things are orderly, with less uncertainty, not more. This has led to an uneasy relationship with entropy in economics, typically ranging from grudging acceptance to outright rejection, as seen from the remarks of two Nobel Laureates in economics, Amartya Sen and Paul Samuelson. Sen observed [38], while commenting on the Theil Index, "given the association of doom with entropy in the context of thermodynamics, it may take a little time to get used to entropy as a good thing ('How grand, entropy is on the increase!'), but it is clear that Theil's ingenious measure has much to be commended. …But the fact remains that it is an arbitrary formula, and the average of the logarithms of the reciprocals of income shares weighted by income shares is not a measure that is exactly overflowing with intuitive sense. It is, however, interesting that the concept of entropy used in the natural sciences can provide a measure of inequality that is not immediately dismissible, however arbitrary it may be". Similar objections were raised by Samuelson [39]: "As will become apparent, I have limited tolerance for the perpetual attempts to fabricate for economics concepts of 'entropy' imported from the physical sciences or constructed by analogy to Clausius–Boltzmann magnitudes". Thus, we run into major conceptual hurdles in the typical statistical mechanics-based approaches to problems in economics, particularly in the study of income and wealth distributions.

Besides these conceptual challenges, there is also a technical one due to the nature of the datasets in economics. As Ormerod [37] and Perline [40] discuss, one can easily misinterpret data from lognormal distributions, particularly from truncated datasets, as inverse power law or other distributions. Therefore, empirical verification of econophysics models is still in the early stages.

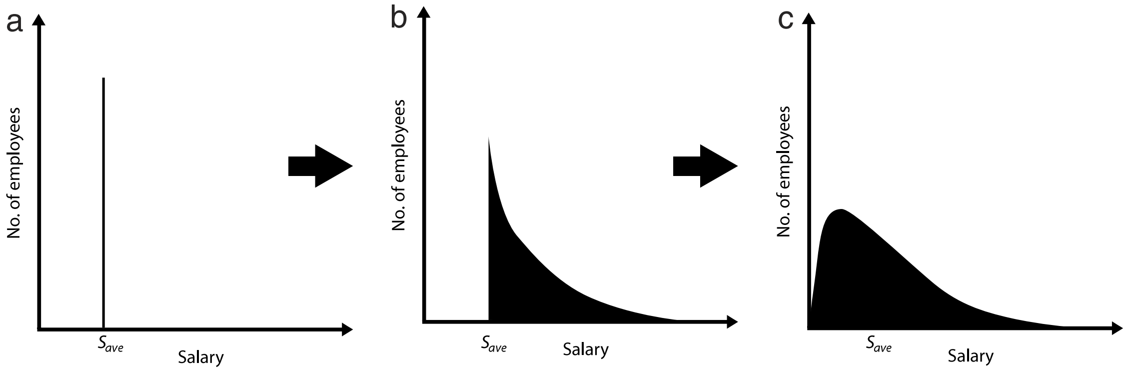


Fig. 1. Spreading of the pay distribution under competition in an ideal free market environment.

Addressing one of the two conceptual challenges, Venkatasubramanian proposed an information-theoretic framework [41,42] wherein he identified that entropy really is a measure of *fairness* in a distribution, not just randomness or uncertainty, which then makes it an appropriate candidate in economics. In this paper, we follow up on this line of inquiry and address the other critical challenge of reconciling the behavior of goal-driven, teleological, agents with that of purpose-free, randomly driven molecules. This resolution helps us address the inequality questions we raised above. We first start from a familiar ground in economics, namely, game theory, to develop a new conceptual microeconomic framework. This leads to surprising and useful insights about a deep connection between game theory and statistical mechanics, paving the way for a general theoretical framework that unifies the dynamics of purposeful animate agents with that of purpose-free inanimate ones.

2. Pay distribution in an ideal free market environment: formulating the problem

We follow Venkatasubramanian's [42] approach in formulating the problem and restate it here for the convenience of the reader. Consider the following *gedankenexperiment* where we study a competitive, dynamic, free market environment comprising of a large number of utility maximizing rational agents as employees and profit maximizing rational agents as corporations. We assume an ideal environment where the market is perfectly competitive, transaction costs are negligible, and no externalities are present. In this ideal free market, employees are free to switch jobs and move between companies in search of better utilities. Similarly, companies are free to fire and hire employees in order to maximize their profits. We do not consider the effect of taxes.

We also assume that a company needs to retain all its employees in order to survive in this competitive market environment. Thus, a company will take whatever steps necessary, allowed by its constraints, to retain all its employees. Similarly, all employees need a utility to survive and that they will do whatever is necessary, allowed by certain norms, to stay employed. We assume that neither the companies nor the employees engage in illegal practices such as fraud, collusion, and so on.

In this ideal free market, consider a company A with N employees and a salary budget of M, with an average salary of $S_{\text{ave}} = M/N$. Let us assume that there are n categories of employees — ranging from secretaries to the CEO, contributing in different ways towards the company's overall success and value creation. All employees in category i contribute value V_i, $i \in \{1, 2, \dots, n\}$, such that $V_1 < V_2 < \cdots < V_n$. Let the corresponding value at S_{ave} be V_{ave}, occurring at category s. Since all employees are contributing unequally, some more some less, they all need to be compensated differently, commensurate with their relative contributions towards the overall value created by the company. Instead, A has an egalitarian policy that all employees are equal and therefore pays all of them the same salary, S_{ave}, irrespective of their contributions. The salary of the CEO is the same as that of an administrative assistant in the mail room. This salary distribution is a sharp vertical line at S_{ave}, as seen in Fig. 1(a), a Kronecker delta function. As noted, while this may seem fair in a social or moral justice sense (this distribution has a Gini coefficient of 0, which according to the Gini measure is the fairest outcome, but more about this later in the paper), clearly it is not in an economic sense. If this were to be the only company in the economic system, or if A is completely isolated from other companies in the economic environment, the employees will be forced to continue to work under these conditions as there is no other choice.

However, in an ideal free market system there are other choices. Therefore, all those employees who contribute more than the average — i.e., those in value categories V_i such that $V_i > V_{\text{ave}}$ (e.g., senior engineers, vice presidents, CEO), who feel that their contributions are not fairly valued and compensated for by A, will therefore be motivated to leave for other companies where they are offered higher salaries. Hence, in order to survive A will be forced to match the salaries offered by others to retain these employees, thereby forcing the distribution to spread to the right of S_{ave}, as seen in Fig. 1(b).

At the same time, the generous compensation paid to all employees in categories V_i such that $V_i < V_{\text{ave}}$, will motivate candidates with the relevant skill sets (e.g., low-level administration, sales and marketing staff) from other companies to compete for these higher paying positions in A. This competition will eventually drive the compensation down for these overpaid employees forcing the distribution to spread to the left of S_{ave}, as seen in Fig. 1(c). Eventually, we will have a distribution that is not a delta function, but a broader one where different employees earn different salaries depending on the values of their contributions as determined by the free market. The funds for the higher salaries now paid to the formerly underpaid employees (i.e., those who satisfy $V_i > V_{\text{ave}}$) come out of the savings resulting from the reduced salaries of the formerly overpaid group (i.e., those who satisfy $V_i < V_{\text{ave}}$), thereby conserving the total salary budget M.

Thus, we see that concerns about *fairness* in pay cause the emergence of a more equitable salary distribution in a free market environment through its self-organizing, adaptive, evolutionary dynamics and that its spread is closely related to fairness in relative compensation. The point of this analysis is not to model the exact details of the free market dynamics but to show that the *notion of fairness* plays a central role in driving the emergence and spread of the salary (in general, utility) distribution through the free market mechanisms.

Even though an individual employee cares only about her utility and no one else's, the collective actions of all the employees, combined with the profit maximizing survival actions of all the companies, in an ideal competitive free market environment of supply and demand for talent, under resource constraints, lead towards a more fairer allocation of pay, guided by Adam Smith's "invisible hand" of self-organization.

We have used salary as a proxy for utility in this example to motivate the problem. In general, utility for an employee is a complicated aggregate that depends on a host of factors, some measurable some not. Obviously, pay (i.e., total compensation including base salary, bonus, options, etc.) is an important component of the utility. Other components include, quantity and quality of the effort, title and peer recognition, competition and job security, career and personal growth opportunities, retirement and health benefits, company culture and work environment, job location, and so on, not necessarily in that order.

Given this free market dynamics scenario, three important questions arise: (i) Will this self-organizing dynamics lead to an equilibrium distribution or will the distribution continually evolve without ever settling down? (ii) If there exists an equilibrium distribution, what is it? (iii) Is this distribution fair?

Our knowledge of the free market dynamics is incomplete, in an important way, without an answer to these fundamental questions. This requires a theoretical understanding of the free market dynamics, at a reasonable level of depth, particularly from the bottom up, agents-based, microeconomic perspective as described above. Given the obvious complexity of this dynamics, it is unrealistic to expect to develop a theory, and the associated models, that will address all the details and nuances. Therefore, our goal is to develop an analytical framework that identifies the key concepts and general principles, models free market dynamics under ideal conditions, and answers these central questions. We propose such a framework in the following sections.

3. A microeconomic game theoretic framework: "restless" agents model

3.1. Formulating the microeconomic payoff function

We address these questions by developing a potential game theoretic microeconomic framework. Continuing with the scenario described above, we assume that all employee agents are generally "dissatisfied" in their current positions, due to aforementioned unfairness considerations. In our model, every employee feels that she is undervalued compared to others in her peer group. Every employee feels she could be doing better, she should be doing better, given her talents and experience, in her company or elsewhere. As a result, they all are constantly on the lookout for job opportunities to improve their utilities. That is, these *utility-maximizing, fairness-seeking, teleological* agents are always *restless*, itching to move.

Even though the overall utility for an employee is a complex aggregate of several factors, as noted above, we hypothesize that, at the minimum, people expect to be compensated fairly for their effort or contribution. They will, of course, accept more compensation if offered, but at the very least they expect fair compensation. Hence we propose that the overall utility is largely determined by three dominant elements: (i) utility from salary, (ii) disutility from effort, and (iii) utility from a fair opportunity for recognition and career advancement. This is the microeconomic foundation on which we build our theory to predict and explain the emergent macroeconomic consequences in pay distribution in an ideal free market environment.

Thus, the overall utility for an agent is given by:

$$h_i(S_i, E_i, N_i) = u_i - v_i + w_i \tag{1}$$

where h_i is the total utility of an employee earning a salary S_i by expending an effort E_i, while competing with $(N_i - 1)$ other agents in the same job category i for a fair recognition of one's contributions. $u(\cdot)$ is the utility derived from salary, $v(\cdot)$ the disutility from effort, and $w(\cdot)$ is the utility from fairness.

The first two elements are rather straightforward to appreciate, but the third requires some more discussion along the lines of the scenario described above. The first two model the tendency of an employee to maximize one's utility from salary while minimizing the effort put into receiving it. As for the third, consider the following. At any job level, an agent is looking to improve her utility only in the jobs space that is *relevant* to her based on her education, experience, and other such qualifications. That is, a receptionist is not eying the job announcement for a CEO. In that sense, what matters in trying to improve one's utility is the *local* competition at the agent's level. It is the assessment of one's *relative* status in a peer group that matters, not its absolute value. For instance, a vice president is not necessarily very happy that she is enjoying much more utility than her receptionist, but is quite unhappy that her peer, another vice president with comparable (or perhaps even less) skills and contributions, has been better recognized in the organization with awards, better work assignments, more perks etc., thereby enjoying a higher utility than her. As far as this "unhappy" agent is concerned, the metric that matters to her is whether she is *one of the chosen few* or *one of the many* in her peer level. Her preference is to be one of the few and possibly the only one enjoying a lot of utility. This is irrespective of the category one is in. The question is not about money, but about a fair valuation, recognition and appreciation of one's abilities and contributions to the organization. These determine her future career prospects in that organization or elsewhere. This is the utility from having a fair shot at a better future.

Now, fairness is a relative quantity, and it arises only when compared with another individual or situation. Consider, for instance, that this employee is eligible for some award. Her chances of winning it goes as $1/N_i$, where N_i is the number of employees in her peer group i, her local competition. Let us further state, to make this line of reasoning clearer, that the award has a monetary equivalent of Q (even though, money is not central to the recognition here). Therefore, her expected value for the award is Q/N_i, and the utility derived from it goes as $\ln(Q/N_i)$ because of diminishing marginal utility.

This leads to utility derived from fairness in recognition and opportunity as:

$$w_i(N_i) = -\gamma \ln N_i. \tag{2}$$

While it is possible to choose other functions which might also capture these properties, we prefer this one for its simplicity and its general use in economics.

For the utility derived from salary, we employ the commonly used logarithmic utility function:

$$u_i(S_i) = \alpha \ln S_i. \tag{3}$$

As for the second element, every job has certain disutility associated with it. This comes from a host of factors such as the investment in education needed to qualify oneself for the job, the experience to be acquired, working hours and schedule, quality of work, work environment, company culture, relocation anxieties, and so on, which are often difficult, if not impossible, to quantify in real life. Hence, it is difficult to capture all these in a single effort metric. While there is considerable prior work on modeling the disutility of effort when a metric for effort is available [43–47], there is not much when such a metric is absent. In the absence of such a metric, typically, one compensates for these different uncertain components of the disutility of a new job by negotiating a salary package that would make it worth the effort. Thus, in practice, one intuitively uses salary as a proxy to impute and estimate the effort involved, thereby estimating the remuneration required to compensate for the disutility of effort. Note that by effort we do not just mean the hours put into performing the job, but all the prior investment in education and experience to qualify oneself as well as the other adjustments and sacrifices one has to make in the new position. There is empirical evidence that supports this line of reasoning in the work of Stratton [48] and Ahituv and Lerman [49] who have demonstrated that effort correlates with $ln(Salary)$. Combining this with the commonly used quadratic disutility from effort [44–47,50–54], we propose the following form for the second element:

$$v_i(E_i) = \beta(\ln S_i)^2. \tag{4}$$

Our formulation is also consistent with the conditions imposed on effort E as a function of salary, $E(S)$. According to Katz [55] and Akerlof and Yellen [56], $E(S)$ should satisfy the following conditions:

$$dE/dS > 0, \qquad E(0) \le 0, \quad \text{and} \quad S/E \times (dE/dS) \quad \text{is decreasing.} \tag{5}$$

Our effort function $E(S) = \ln S$ satisfies all three conditions:

1. $dE/dS = 1/S > 0$
2. $E(0) = -\infty < 0$
3. $S/E \times (dE/dS) = (S/\ln S) \times (1/S) = 1/\ln S$ is decreasing.

Intuitively, one can combine u and v to compute $u_{\text{net}} = au - bv$ (a and b are positive constant parameters) which is the net benefit derived from a job after accounting for its cost. Typically, net benefit will keep increasing as u increases (because of salary increases, for example). However, generally, after a point, the cost has gone up so much because of personal sacrifices such as working overtime, missing quality time with family, giving up on hobbies, job stress resulting in poor mental and physical health, etc., u_{net} begins to decrease after reaching a maximum. Hence, the simplest model of this typical profile is a quadratic function, as in $u_{\text{net}} = au - bu^2$. Since, $u \sim \ln(Salary)$, we get Eq. (4). Therefore, our formulation for effort is a reasonable one supported by empirical evidence as well as by intuitive and theoretical expectations.

Combining all three, we have

$$h_i(S_i, E_i, N_i) = \alpha \ln S_i - \beta(\ln S_i)^2 - \gamma \ln N_i \tag{6}$$

where $\alpha, \beta, \gamma > 0$.

In general, α, β and γ, which model the relative importance an agent assigns to these three elements, can vary from agent to agent. However, we first examine the simple and ideal situation where all agents have the same preferences and hence treat these as constant parameters. (In Section 3.4.2 we relax this requirement and consider other cases.) As noted, presumably, there are other expressions one could use to model these three elements, but the choices we have made have interesting properties, revealing important insights and connections as we shall see shortly.

In order to move to a job with better utility, an agent needs job offers. So, the employee agents constantly gather information and scout the market, and their own companies, for job openings that are commensurate with their skill sets, experiences and career and personal goals. Similarly, the company agents (through their human resources department, for example) also conduct similar searches looking for opportunities to fire and hire employees so that their profits may be improved.

At any given time, an employee agent is faced with one of five job options: (i) no new job offer is available, (ii) new offer has the *same* utility as the current one, (iii) new offer has *less* utility than the current one, (iv) new offer has *more* utility, or (v) is let go from the current job (i.e., *zero* utility). The agent's best strategies for the five options are: for (i), (ii) and (iii), the agent stays put in the current position at the current utility, for (iv) accept the new offer, and (v) leave the company and look for a new position. Each agent's strategy is independent of what the other agents are doing.

We are now ready to answer the first question.

3.2. Is there an equilibrium distribution?

In a potential game framework, payoff is the gradient of potential $\phi(\mathbf{x})$, i.e.,

$$h_i(\mathbf{x}) \equiv \partial\phi(\mathbf{x})/\partial x_i \tag{7}$$

where $x_i = N_i/N$ and $\mathbf{x}$ is the population vector. Therefore, by integration (we replace partial derivative with total derivative because $h_i(\mathbf{x})$ can be reduced to $h_i(x_i)$ expressed in Eqs. (1)–(4)),

$$\phi(\mathbf{x}) = \sum_{i=1}^{n} \int h_i(\mathbf{x}) dx_i \tag{8}$$

We observe that the game under consideration is a potential game with the potential function:

$$\phi(\mathbf{x}) = \phi_u + \phi_v + \phi_w + \text{constant} \tag{9}$$

where

$$\phi_u = \alpha \sum_{i=1}^{n} x_i \ln S_i \tag{10}$$

$$\phi_v = -\beta \sum_{i=1}^{n} x_i (\ln S_i)^2 \tag{11}$$

$$\phi_w = \frac{\gamma}{N} \ln \frac{N!}{\prod_{i=1}^{n} (Nx_i)!} \tag{12}$$

where we have used Stirling's approximation in Eq. (12).

We can show that $\phi(\mathbf{x})$ is strictly concave:

$$\partial^2\phi(\mathbf{x})/\partial x_i^2 = -\gamma/x_i < 0. \tag{13}$$

Therefore, a unique Nash equilibrium for this game exists, where $\phi(\mathbf{x})$ is maximized, as per the well-known theorem [57, p. 60].

It is important to note that this is a stable equilibrium as long as the evolutionary dynamics satisfies positive correlation (e.g., replicator dynamics, Smith dynamics, best response dynamics, etc.), for the potential is a Lyapunov function under such condition, with a guarantee of global convergence [57, p. 223].

This answers our first question.

3.3. Connection with statistical mechanics

Readers familiar with statistical mechanics will recognize the potential component ϕ_w as entropy (except for the missing Boltzmann constant k), and that maximizing the payoff potential in game theoretic equilibrium would correspond to maximizing entropy in statistical mechanical equilibrium, revealing a deep and useful connection between these seemingly different conceptual frameworks. This connection suggests that one may view the statistical mechanics approach to molecular behavior, also called *statistical thermodynamics*, from a potential game perspective. In this approach, one may view the molecules as restless agents in a game (let us call it the *thermodynamic game*), continually jumping from one energy state to another through intermolecular collisions. However, unlike employees who are continually driven to switch jobs in search of better utilities they desire, molecules are *not teleological, i.e., not goal-driven*, in their constant search. As prisoners of Newton's Laws, constantly subjected to intermolecular collisions, their search and dynamical evolution is the result of thermal agitation.

3.4. What is the equilibrium distribution?

This connection to statistical thermodynamics, and the insight that ϕ_w is *entropy* in this context, helps us in answering the second question: What is the equilibrium distribution?

We first answer this question for the thermodynamic game. Approaching the thermodynamic game from potential game perspective, we have the following "utility" for molecules in state i:

$$h_i(E_i, N_i) = -\beta E_i - \ln N_i \tag{14}$$

where E_i is the energy of a molecule in state i (not to be confused with effort in (4)), $\beta = 1/kT$, $k = 1.38 \times 10^{-23}$ JK^{-1} is the Boltzmann constant; and T is temperature. By integrating the utility, we can obtain the potential of the thermodynamic game:

$$\phi(\mathbf{x}) = -\frac{\beta}{N}E + \frac{1}{N}\ln\frac{N!}{\prod_{i=1}^{n}(Nx_i)!} \tag{15}$$

where $E = N\sum_{i=1}^{n} x_i E_i$ is the total energy that is conserved.

We use the method of Lagrange multipliers with L as the Lagrangian and λ as the Lagrange multiplier for the constraint $\sum_{i=1}^{n} x_i = 1$:

$$L = \phi + \lambda(1 - \sum_{i=1}^{n} x_i). \tag{16}$$

Solving $\partial L/\partial x_i = 0$ and substituting the results back to $\sum_{i=1}^{n} x_i = 1$, we obtain the well-known Gibbs–Boltzmann exponential distribution at equilibrium:

$$x_i = \frac{\exp(-\beta E_i)}{\sum_{j=1}^{n}\exp(-\beta E_j)}. \tag{17}$$

What we just now did is the standard procedure followed in maximum entropy methods in statistical mechanics and information theory to identify the distribution that maximizes entropy under the given constraints [58–60]. Once again, readers familiar with statistical thermodynamics will recognize that from (15), we have:

$$\phi = -\frac{1}{NkT}(E - TS) = -\frac{\beta}{N}A \tag{18}$$

where A is the Helmholtz free energy, S is entropy (not to be confused with salary), and T is temperature. Indeed, in statistical mechanics A is called a thermodynamic potential.

For the teleodynamic game, i.e., the pay distribution game, we carry out the same procedure to maximize $\phi(\mathbf{x})$ in Eqs. (9)–(12) to obtain the following lognormal distribution at equilibrium:

$$X_i = \frac{1}{S_i D}\exp\left[-\frac{\left(\ln S_i - \frac{\alpha+\gamma}{2\beta}\right)^2}{\gamma/\beta}\right] \tag{19}$$

where $D = N\exp\left[\lambda/\gamma - (\alpha+\gamma)^2/4\beta\gamma\right]$ and λ is the Lagrange multiplier.

3.4.1. Replicator dynamics

Alternatively, we can approach this question from the replicator dynamics point of view in game theory [57]. In this approach, an agent revises its strategy based on

$$\rho_{ij} \propto x_j[h_j - h_i]_+. \tag{20}$$

Under this protocol, an agent in the job category i who receives a revision opportunity, i.e., a new job offer in category j, switches from i to j with probability ρ_{ij}. Therefore the dynamics becomes:

$$\dot{x}_i \propto x_i\left(h_i - \sum_{j=1}^{n} x_j h_j\right). \tag{21}$$

The equilibrium is reached (i.e., $\dot{x} = 0$) when individual payoff equals the average payoff of the system:

$$h_i^* = \sum_{j=1}^{n} x_j h_j^* = h^*. \tag{22}$$

We ignore the trivial solution of $x_i = 0$. Substituting this equation back in our utility function (Eq. (1)), we solve to find the equilibrium distribution to be

$$X_i = \frac{1}{S_i D} \exp\left[-\frac{\left(\ln S_i - \frac{\alpha+\gamma}{2\beta} \right)^2}{\gamma/\beta} \right] \tag{23}$$

where $D = N \exp\left[h^*/\gamma - (\alpha+\gamma)^2/4\beta\gamma\right]$. This result agrees with (19). The equilibrium average payoff is therefore

$$h^* = \gamma \ln Z - \gamma \ln N \tag{24}$$

where $Z = \sum_{j=1}^{n} \exp\{[\alpha \ln S_j - \beta(\ln S_j)^2]/\gamma\}$ resembles the partition function seen in statistical mechanics.

This result is also in agreement with what Venkatasubramanian [41,42] derived using an information theoretic framework. In that approach, the constraints are determined by information typically known about the distribution *a priori*. They are: (i) total number of employees N, (ii) total amount of money M budgeted to pay all these employees, (iii) minimum salary, $S_{\min}$, received by the lowest paid employee, often fixed by the minimum wage law or a reservation wage, and (iv) the maximum salary, $S_{\max}$, cannot exceed M. As Venkatasubramanian has shown, maximizing entropy under these constraints leads to a lognormal distribution at equilibrium given by:

$$f(S; \mu, \sigma) = \frac{1}{S\sigma\sqrt{2\pi}} \exp\left[-\frac{(\ln S - \mu)^2}{2\sigma^2} \right] \tag{25}$$

where $\mu = \ln(M/N) - \sigma^2/2$; $\sigma = (\ln M - \ln S_{\min})/2a$; and a is a parameter chosen using the Chebyshev inequality given by:

$$\text{Prob}(-a\sigma < X - \mu < a\sigma) \geq 1 - \frac{1}{a^2} \tag{26}$$

to the level of confidence desired in the estimate for σ (e.g. for $a = 10$, $P \geq 0.99$). Eq. (25) is the same as (19) or (23) with the following identities:

$$\begin{cases} \mu = \dfrac{\alpha+\gamma}{2\beta} \\ \sigma = \left(\dfrac{\gamma}{2\beta} \right)^{1/2}. \end{cases} \tag{27}$$

For the thermodynamic game, it is easy to show from Eqs. (20) through (22), and (14), a similar replicator dynamics analysis produces the same Gibbs–Boltzmann exponential distribution in (17) at equilibrium.

Thus, we see that, intuitively, maximizing the game theoretic potential (Eq. (9) or (15)) is the same as maximizing entropy subject to the constraints. In the statistical mechanical or information theoretic formulations, these constraints are separately imposed on entropy whereas in the game theoretic formulation (Eq. (9) or (15)) the constraints are already embedded in the equation (the only additional constraint imposed is the total number of agents, N). Therefore, the resulting Lagrangian (e.g., Eq. (16)) is the same, thereby leading to the same distribution. These demonstrate the internal consistency among the three different approaches, namely, potential game theory, replicator dynamics, and statistical mechanics, which is reassuring.

3.4.2. A bi-population game

Our model with a homogeneous population of agents with the same payoff preferences describes a 1-class system (i.e., a class-less organization or society), a model of an ideal, utopian, organization or society. This utopian system recognizes the dignity of labor, and that every agent is important, making a valuable contribution. The other extreme would be one where different agents have different payoff preferences, i.e., different α, β, and γ values. However, reality is typically somewhere in between, with Π different classes of agents, with all the agents in the same class having the same α, β, and γ values. For instance, employees in an organization, or in a society at large, are often grouped into three broad classes — e.g., *["blue collar", "white collar", c-suite executives or owners]* or *[lower, middle and upper class]*. Such coarse-grained classification is often appropriate, even necessary sometimes, to elicit and discern macroscopic trends in a population. With that in mind, we now present the analysis for a 2-class (i.e., bi-population) system containing two classes of agents each with a distinct set of α, β, and γ values. We then show how this can be generalized to Π-class systems.

The utility of an agent in a 2-class system at salary level i is therefore defined as

$$h_{i,j} = \alpha_j \ln S_i - \beta_j (\ln S_i)^2 - \gamma_j \ln(N_{i,1} + N_{i,2}) \tag{28}$$

where the choice of $j \in \{1, 2\}$ indicates either Class 1 population or Class 2 population.

The equilibrium replicator dynamics is also modified:

$$\begin{cases} h_{i,j}^* = h_j^* & \forall i \in \Omega_j \\ h_{k,j} < h_j^* & \forall k \notin \Omega_j \end{cases} \tag{29}$$

where $\Omega_j = \{k | x_{k,j}^* > 0\}$ denotes the collection of levels with class j's presence. The first condition is identical to the homogeneous scenario. The second indicates the possibility that some levels are only occupied by a single class (i.e., the utility is too low for the other class).

We can prove that $\Omega_1 \cup \Omega_2 = \{k | 1 \le k \le n\}$ and $\Omega_1 \cap \Omega_2 = \emptyset$, i.e., every salary level contains some population but not both. First, suppose there are empty salary levels. They will soon be occupied because of the infinitely high utilities. Thus Ω_1 and Ω_2 cover the whole domain. Second, suppose there is an overlap where $\hat{\Omega} = \Omega_1 \cap \Omega_2$. Let the equilibrium population density be x^*. Eq. (29) can be rewritten as:

$$h_j^* = \alpha_j \ln S_i - \beta_j (\ln S_i)^2 - \gamma_j \ln N x^* \quad \forall i \in \hat{\Omega}. \tag{30}$$

Thus

$$\lim_{\Delta S \to 0} \frac{\Delta x^*}{\Delta S_i} = \frac{dx^*}{dS} = \frac{\alpha_j - 2\beta_j \ln S}{\gamma_j S} x^*. \tag{31}$$

This indicates two distinct gradients for *every* point in $\hat{\Omega}$. Therefore we will not see an overlapping region with mixed populations.

We can also prove that the equilibrium density curve is continuous at the interface of two populations. Suppose otherwise, at the interface $S = \hat{S}$,

$$x_1^*(\hat{S}) \neq x_2^*(\hat{S}) \tag{32}$$

according to Eq. (29) again,

$$\alpha_j \ln \hat{S} - \beta_j (\ln \hat{S})^2 - \gamma_j \ln x_j^* \ge \alpha_j \ln \hat{S} - \beta_j (\ln \hat{S})^2 - \gamma_j \ln x_{-j}^* \tag{33}$$

i.e.,

$$x_j^* \ge x_{-j}^*. \tag{34}$$

The only possible solution is $x_1^* = x_2^*$ therefore the population density is continuous at the interface.

Even though we now know these equilibrium characteristics of a bi-population game, an exact equilibrium density is still tedious to obtain unless Ω_1 and Ω_2 are given:

$$x_i = \frac{N_1/N}{S_i D_1} \exp\left[-\frac{\left(\ln S_i - \frac{\alpha_1+\gamma_1}{2\beta_1}\right)^2}{\gamma_1/\beta_1} \right] \mathbb{1}(i \in \Omega_1) + \frac{N_2/N}{S_i D_2} \exp\left[-\frac{\left(\ln S_i - \frac{\alpha_2+\gamma_2}{2\beta_2}\right)^2}{\gamma_2/\beta_2} \right] \mathbb{1}(i \in \Omega_2) \tag{35}$$

where D_j is the normalization that ensures $\sum_{i=1}^n x_i = 1$. We can, however, get a good approximation when the two lognormal curves of Class 1 and Class 2 are sufficiently separated such that the overlap is insignificant. The overall distribution is then estimated as a mixture of two lognormal distributions:

$$X_i \approx \frac{N_1}{S_i D} \exp\left[-\frac{\left(\ln S_i - \frac{\alpha_1+\gamma_1}{2\beta_1}\right)^2}{\gamma_1/\beta_1} \right] + \frac{N_2}{S_i D} \exp\left[-\frac{\left(\ln S_i - \frac{\alpha_2+\gamma_2}{2\beta_2}\right)^2}{\gamma_2/\beta_2} \right] \tag{36}$$

where N_j denotes the number of class j agents and D denotes the normalization parameter which is easily computed.

To test the model predictions, we ran an agent-based simulation comprising of one million agents in two classes, at 100 salary levels, with a minimum pay of \$20,000 (using a minimum wage of \$10/hr and 2000 h/year) and a maximum pay of \$3,000,000. We explored the typical case of 95% of the population in Class 1 and 5% in Class 2. The respective α, β, and γ values for the two classes are shown in Table 1. The dynamics unfolds by having each agent trying to maximize its utility given by (6) by switching from its current job to a better one and the equilibrium (stationary) distribution emerges over time, as shown in Figs. 2 and 3. In Fig. 2, the blue (Class 1) and yellow (Class 2) histogram bars are data from the simulation and the lines are predictions by the model. As the results show, the two populations are sufficiently separated and hence the individual lognormal distributions predicted by the model (Eq. (36)) fit the data very well. For the population shown in yellow (Class 2), its higher α makes it value the utility from salary more, lower β motivates it to put in more effort, and higher γ makes the utility from fairness more important, compared to the Class 1 agents. As a result, Class 2 agents are averse to jobs with lower pay. It is the opposite for the agents from Class 1. We observe that the combined distribution (solid red line), as one might expect, fits the lognormal distribution for the blue population (Class 1) quite well in the lower and medium salary ranges but deviates from it for higher salaries.

Table 1
2-class system parameters.

j	α_j	β_j	γ_j
1	93.4	3.87	2.17
2	95.8	3.67	4.34

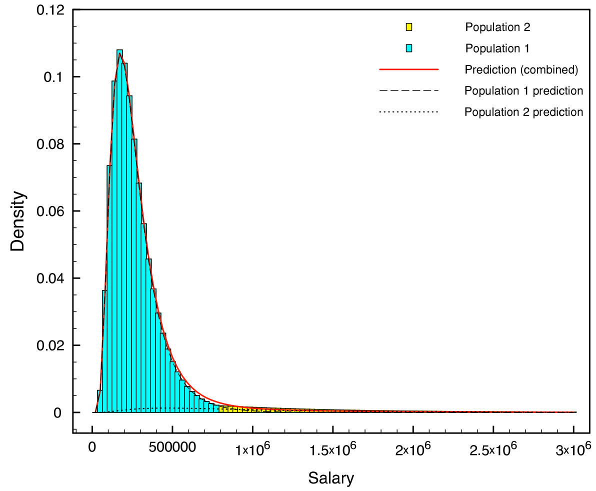


Fig. 2. Simulation results.

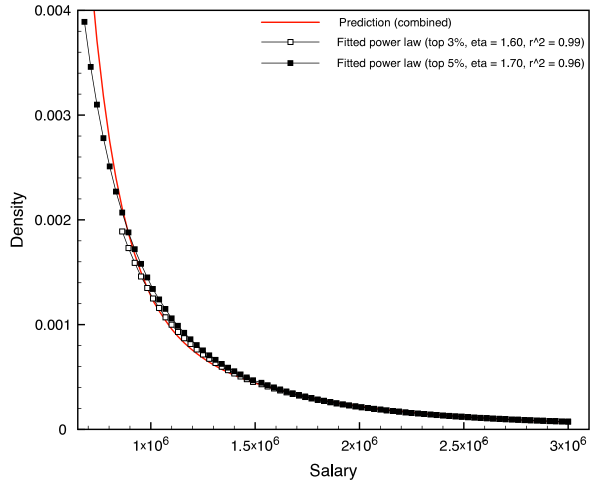


Fig. 3. Fitted power law.

We also show that the distribution of the higher salaries (largely occupied by the yellow population agents) can be fitted to an inverse power law, given as follows:

$$x_i \propto S_i^{-(1+\eta)} \tag{37}$$

1. Top 3% fitted: $\eta = 1.60$, $r^2 = 0.99$
2. Top 5% fitted: $\eta = 1.70$, $r^2 = 0.96$.

We see that the inverse power law fit is very good for both top 3% and 5%. The Pareto exponents from our simulation data agree well with empirical data reported in the literature — between 1 and 2, but typically around 1.5 for the top 3% [33]. Thus, the main lesson here is that while the overall distribution is a combination of two lognormal distributions, it can be quite easily misidentified as a lognormal for the majority and an inverse power law or Pareto distribution for the minority at the top end of the salaries. This again confirms similar warnings by Perline [40] and Mitzenmacher [61]. For actual salary distributions reported in the literature [33], the available data is not good enough to sort this out clearly and further studies are needed.

Our 2-class approach can be generalized for a Π-class game along the lines we described above. However, as noted, we suggest that we might need only three classes, at most four, to model empirical data effectively. At any rate, at the present time, the empirical data reported in the literature is not good enough to test 3-class or 4-class models. The best it seems to be able to do is to identify the need for a 2-class model, but even there it appears unable to discriminate between a lognormal distribution and a power law fit for the top 3%–5% as we showed above.

An interesting question one might ask is "Why does the 2-class split in actual data occur at about 95%–97% of the population? Why not at 80%, for instance?" In our theory, this is related to the fraction of the population that is highly motivated, talented, and driven towards individual accomplishments and success (i.e., Class 2). It would be nice if we had demographic data that directly showed where this 2-class split occurs in the real world, but we do not. One approximation we can perhaps use to estimate how human abilities are distributed in a population is how IQ is distributed in a population. Obviously, as we all know, IQ does not capture the complete picture of the human talent spectrum and how people succeed. Nevertheless, it is interesting to note that 2-σ deviation (IQ $= \sim$130) from the median IQ value occurs at $\sim$97% of the population — i.e., the top $\sim$3% of the population have an IQ greater than $\sim$130, beyond the 2-σ deviation.

3.5. *Is the equilibrium distribution fair?*

The fairness question is a challenging one as the term fairness is frequently used quite broadly and can mean different notions in different contexts, as can be seen from the extensive literature on this subject (we cite here only a few selected papers [62–66,38,67] as a representative sample). This is one of the reasons the term is often used within quotes, as in "fair". As we saw in Section 2, fairness based on moral principles would require us to recognize all human beings as equals, but does this imply that everyone should receive equal pay irrespective of their contributions in an organization? Our sense of economic fairness would disagree. Our intuitive sense of fairness in pay suggests that one's reward should be commensurate with the value of one's contribution.

Even within the economic context, there are a variety of measures of fairness in use. For instance, given an unequal distribution of pay among employees in an organization, some commonly used measures are maximin fairness, proportional fairness, Gini coefficient, Theil index and so on [41,42]. For most fairness measures, the implementation of fairness in a system requires a *central authority* to care about the weakest agent(s) in the system and promote fairness by enforcing a fairness policy utilizing the chosen measure. For example, in the Rawlsian framework, the state, which acts as the central authority, is needed to promote fairness by enforcing the maximin measure based fairness policy. However, in an ideal free market environment there is *no such central authority*. Each economic agent is interested in maximizing only his or her utility or profit and does not care about anyone else's. So, does such a free market environment care about fairness in pay distribution? As we showed, the answer is yes. The ideal free market does promote fairness, as an emergent property resulting from the self-organizing dynamics of the market environment.

As we showed, the deep connection between game theory and statistical mechanics, and with information theory, occurs via entropy, a concept that is often misunderstood and much maligned [41,42,39,38]. In the past, there have been several attempts to find a suitable interpretation of entropy for economic systems without much success [68–70,39]. In these attempts, one typically wrote down equations in economics that mimicked expressions in thermodynamics for entropy, energy, temperature, etc. — but no identification of entropy in terms of meaningful economic concepts was made. Just as entropy is a measure of disorder in thermodynamics and uncertainty in information theory, what does entropy mean in economics? Neither interpretation, disorder or uncertainty, makes much sense in the economic context. Economic systems work best when they have orderly markets. Why then would anyone want to maximize disorder? Similarly, economic systems work best when there is less uncertainty. Why then would anyone want to maximize uncertainty? The inability to resolve this crucial issue has been a major conceptual roadblock for decades thwarting meaningful progress, as evidenced from Amartya Sen's remarks about the Theil index [38] or Paul Samuelson's objections to entropy in economics [39].

The crucial insight here is the recognition that entropy is a measure of *fairness* in a distribution, an insight that has not been explicitly recognized and particularly stressed in prior work in statistical thermodynamics, information theory, or economics [42]. Despite the several attempts in the past, entropy has played, by and large, only a marginal role in economics, even that with strong objections from leading practitioners. Its pivotal role in economics and in free market dynamics has never been recognized. This is mainly because entropy's essence as fairness appears as different facets in different contexts [42]. In thermodynamics, being *fair* to all accessible phase space cells at equilibrium under the given constraints — i.e., assigning *equal* probabilities to all the allowed microstates — projects entropy as a measure of *randomness or disorder* [71]. This is the appropriate interpretation in this particular context, but it obscures the essential meaning of entropy as a measure of fairness. In information theory, being *fair* to all messages that could potentially be transmitted in a communication channel — i.e., assigning equal probabilities to all the messages — shows entropy as a measure of *uncertainty* [58,59]. Again, while this is the appropriate interpretation for this application, this, too, conceals the real nature of entropy. In the design of teleological systems, being fair to all potential operating environments, entropy emerges as a measure of *robustness* i.e., maximizing system safety or minimizing risk [72]. Once again, this is the right interpretation for this domain, but this also hides its true meaning.

Thus, the common theme across all these different contexts is the essence of entropy as a measure of fairness, which stems from the notion of *equality* expressed mathematically. Principles of *equality* and *proportionality* are the foundations of our sense of fairness — e.g., equal pay for equal work (i.e., contribution) and more pay for more work. If there are N possible candidates among whom a resource is to be distributed, and if no particular candidate is to be preferred over another, then the fairest distribution of the resource is one of *equal* allocation among all of them. This quantitative mathematical relationship is at the core of the concept of fairness. Bernoulli and Laplace expressed this notion in probability theory as the *Principle of Insufficient Reason*. The generalization of this principle is the *Principle of Maximum Entropy* [58] which addresses the question: "What is the fairest assignment of probabilities of several alternatives given a set of constraints?" Thus, the roots of entropy as a fairness measure can be traced all the way back to the *Principle of Insufficient Reason* [42]. Somehow, this important insight seems to have been missed in all these years since the discovery of entropy.

It is a historical accident that the concept of entropy was discovered in the context of thermodynamics and, therefore, unfortunately, got tainted with the negative notions of doom and gloom, while, ironically, it really is a measure of fairness, which is a good thing. Even its subsequent "rediscovery" by Shannon in the context of information theory did not help much, as entropy now got associated with uncertainty, again not a good thing. Oxford English Dictionary (OED) defines entropy as "lack of order or predictability; gradual decline into disorder: e.g., 'a marketplace where entropy reigns supreme'". Such definitions only reaffirm the common interpretation of entropy as disorder and uncertainty. But, as we show in this paper, OED was more correct, ironically, than they knew, in their example sentence 'a marketplace where entropy reigns supreme', though not in the way they meant it. Indeed, in the free market entropy does reign supreme!

It is important not to confuse entropy as a concept from physics even though it was first discovered there. In other words, it is not like energy or momentum, which are physics-based concepts. Entropy really is a concept in probability and statistics, an important property of distributions, whose application has been found to be useful in physics and information theory. In this regard, it is more like variance which is a property of distributions, a statistical property, with applications in a wide variety of domains. However, as a result of this profound, but understandable, confusion about entropy as a physical principle, one got trapped in the popular notions of entropy as randomness, disorder, doom or uncertainty, which has prevented people from seeing the deep and intimate connection between statistical theories of inanimate systems composed of non-rational entities (e.g., gas molecules in thermodynamics) and of animate, teleological, systems of rational agents seen in biology, economics, and sociology.

In addition, and most crucially for economics, entropy's connection with the self-organizing free market dynamics has not been made before. Our contribution demonstrates that the ideal free market for labor promotes fairness as an emergent self-organized property and identifies entropy as the appropriate measure of this fairness. We believe that by properly recognizing entropy as a measure of fairness, a fundamental economic and social principle, and showing how it is naturally and intimately connected to the dynamics of the free market, our theory makes a significant conceptual advance in revealing the deep and direct connections between game theory, statistical thermodynamics, information theory, and economics.

This revelation of entropy's true meaning also sheds new light on a decades-old fundamental question in economics, as Samuelson [73] posed in his Nobel Lecture, "what it is that Adam Smith's 'invisible hand' is supposed to be maximizing", or as Monderer and Shapley [74] stated regarding the potential function P^* in game theory, "This raises the natural question about the economic content (or interpretation) of P^*: What do the firms try to jointly maximize? We do not have an answer to this question".

Our theory suggests that what all the agents in a free market environment are jointly maximizing, i.e., what the "invisible hand" is maximizing, is *fairness*. Maximizing entropy, or game theoretic potential, is the same as maximizing fairness *collectively* in economic systems, i.e., being fair to everyone under the given constraints. In other words, economic equilibrium is reached when every agent feels she or he has been fairly compensated for her or his efforts. As we all know, fairness is a fundamental economic principle that lies at the foundation of the free market system. It is so vital to the proper functioning of the markets that we have regulations and watchdog agencies that break up and punish unfair practices such as monopolies, collusion, and insider trading. Thus, it is eminently reasonable, indeed particularly reassuring, to find that maximizing fairness collectively, i.e., maximizing entropy, is the condition for achieving economic equilibrium. We call this result the *fair market hypothesis*. We claim that the ideal free market, in addition to being efficient, also promotes fairness to the maximum level allowed by the constraints imposed on it. A related interpretation is that the game theoretic potential captures the trade offs among utility from salary, disutility from effort and utility from fairness, for all the agents collectively. The ideal free market tries to accommodate every agent's individual preference regarding this trade off, given the overall constraints on money and job openings. Thus, in a sense, the market is trying to maximize "harmony", an accord freely and jointly agreed to by all the agents, where every agent feels fairly compensated for his or her effort.

In summary, for a homogeneous (i.e., 1-class) population, the maximum entropy distribution, namely, the lognormal distribution, resulting from free market dynamics, is the fairest distribution of pay in a large organization. It is the fairest inequality of pay in an ideal free market society.

3.6. Global trends in income inequality: model vs reality

We now compare our theory's predictions with real-world data on income distributions from different countries. While our theory is developed for modeling pay distributions in large corporations, its predictions may be compared with country-wide pretax income (excluding capital gains) data as most of it is an aggregate of the pay of individuals. In essence, we are approximating an entire country as a large corporation functioning in a free market environment. We compare our model's predictions for the shares of the total income by three segments of the population (namely, bottom 90%, top 10%–1% and top 1%) with those observed in different countries as reported by Piketty and his colleagues in their World Top Incomes (WTI)

Table 2
Model predictions and sample empirical data for different countries.

Country	Norway	Sweden	Denmark	Switzerland	Netherlands	Australia	France	Germany	Japan	Canada	UK	US
Year	2011	2012	2008	2010	1999	2010	2006	2008	2010	2000	2011	2013
Currency	NOK	SEK	DKK	CHF	NLG	AUD	EUR	EUR	K JPY	CAD	GBP	USD
Minimum	139,152	159,630	146,919	37,266	28,142	29,716	15,885	17,927	1,236	11,036	11,324	15,131
Average	319,391	249,760	202,502	67,056	63,557	49,304	26,872	29,826	2,255	24,859	19,217	52,619
Maximum	3,874,714	2,580,353	1,747,643	1,047,750	416,931	688,700	366,769	630,374	32,608	528,156	365,130	1,424,300
μ	12.52	12.32	12.13	10.96	10.96	10.67	10.06	10.13	7.57	9.91	9.70	10.58
σ	0.55	0.46	0.41	0.56	0.45	0.52	0.52	0.59	0.55	0.64	0.58	0.76
Gini	0.30	0.26	0.23	0.31	0.25	0.29	0.29	0.33	0.30	0.35	0.32	0.41
Bottom 90% ideal	76.6%	79.3%	80.8%	76.6%	79.7%	77.6%	77.6%	75.4%	76.9%	73.8%	75.9%	70.0%
Top 10%–1% ideal	19.5%	17.5%	16.5%	19.6%	17.2%	18.9%	18.8%	20.4%	19.3%	21.6%	20.1%	24.2%
Top 1% ideal	3.8%	3.1%	2.8%	3.8%	3.0%	3.6%	3.6%	4.2%	3.7%	4.6%	4.0%	5.8%
Bottom 90% data	71.7%	72.1%	73.8%	66.5%	71.9%	69.0%	67.2%	60.5%	59.5%	58.9%	60.9%	53.0%
Top 10%–1% data	20.5%	20.8%	20.1%	22.9%	22.7%	21.8%	23.9%	25.6%	31.0%	28.0%	26.2%	29.5%
Top 1% data	7.8%	7.1%	6.1%	10.6%	5.4%	9.2%	8.9%	13.9%	9.5%	13.2%	12.9%	17.5%
Bottom 90% ψ	−6.5%	−9.1%	−8.6%	−13.2%	−9.8%	−11.0%	−13.4%	−19.8%	−22.6%	−20.2%	−19.8%	−24.3%
Top 10%–1% ψ	5.1%	18.4%	22.2%	16.7%	31.7%	15.7%	26.7%	25.6%	60.3%	29.7%	30.5%	21.9%
Top 1% ψ	104.2%	128.1%	117.4%	177.3%	77.8%	156.6%	150.5%	234.3%	153.8%	184.3%	221.0%	200.7%

Database [75]. While it would be more accurate to use the 2-class version of our model for the comparison, as it represents the top ∼3% better, it is difficult to determine uniquely the parameters which define where the first population ends and the second begins. Therefore, we use the 1-class model as the ideal reference, which predicts a single lognormal distribution for the entire population. We fully expect real-life free market societies to deviate from ideality, but we are interested in understanding how big the deviations are and why. While the income distributions should be lognormal in the different countries, if they were functioning as ideal free market societies, they may have different means and variances depending on the minimum, average and maximum annual income in the different countries, which determine the μ and σ of the corresponding lognormal distributions. As an example, we show these parameters in Table 2 for the different countries.

The minimum, average and maximum income data are obtained from the WTI Database. For the maximum income, we chose to use the threshold income at 99.9%. At this cutoff, the area under the lognormal curve would correspond to 6σ. Thus, we estimate σ by using the approximation

$$6\sigma \approx \ln(\mathit{Maximum\ income}) - \ln(\mathit{Minimum\ income}) \tag{38}$$

$$\mu = \ln(\mathit{Average\ income}) - \frac{\sigma^2}{2}. \tag{39}$$

For UK and Netherlands where the 99.9% threshold data is not available, we found that it is well approximated by the average salary of the top 0.5%, by testing this heuristic for the other countries where the threshold is known. For Switzerland, Sweden, Norway and Denmark, there is no minimum wage requirement and hence that data is not available. For these countries, we consulted several country-specific sources to obtain guidelines about what the typical minimum wage-like compensation might be for entry level positions in recent years (2010–2014). We then used historic data on annual increases for the average income of the bottom 90% of the population to deflate and back calculate the minimum wage for the past years.

Once μ and σ are known, we can uniquely determine the corresponding lognormal distribution, and compute the income shares of the bottom 90%, top 10%–1%, and the top 1%. Since μ and σ typically vary from year to year (because of the changes in the minimum, average, and/or maximum income from year to year), the ideal distribution of income to the bottom 90%, top 10%–1%, and the top 1%, as predicted by the model, also varies from year to year, though not by much. As an example, these values are displayed in Table 2 for the twelve countries (which are commonly used examples [8]), for the years shown.

Now, we know from empirical data that the free market economies of the Scandinavian countries are generally more fair in the economic treatment of all their citizens, not just the wealthy ones. We also know that US does not do as well. We further know that other Western European countries such as France, Germany, and Switzerland, are somewhere in between.

Can the model predict these outcomes? That is, just by knowing only the minimum, average and maximum incomes in a free market society, can the model identify how fair these societies are?

To test the model along these lines, we define a new index of inequality that uses the ideal lognormal distribution (one that is appropriate for the country under consideration) as the reference. This new measure, called *Nonideal Inequality Coefficient*, ψ, is defined as:

$$\psi = \frac{\mathit{Actual\ share}}{\mathit{Ideal\ share}} - 100\%. \tag{40}$$

ψ measures the level of nonideal inequality in the system. When ψ is zero, the system has the ideal level of inequality, the fairest inequality. When ψ is small, the level of inequality is almost ideally fair; when it is large, the inequality is more unfair.

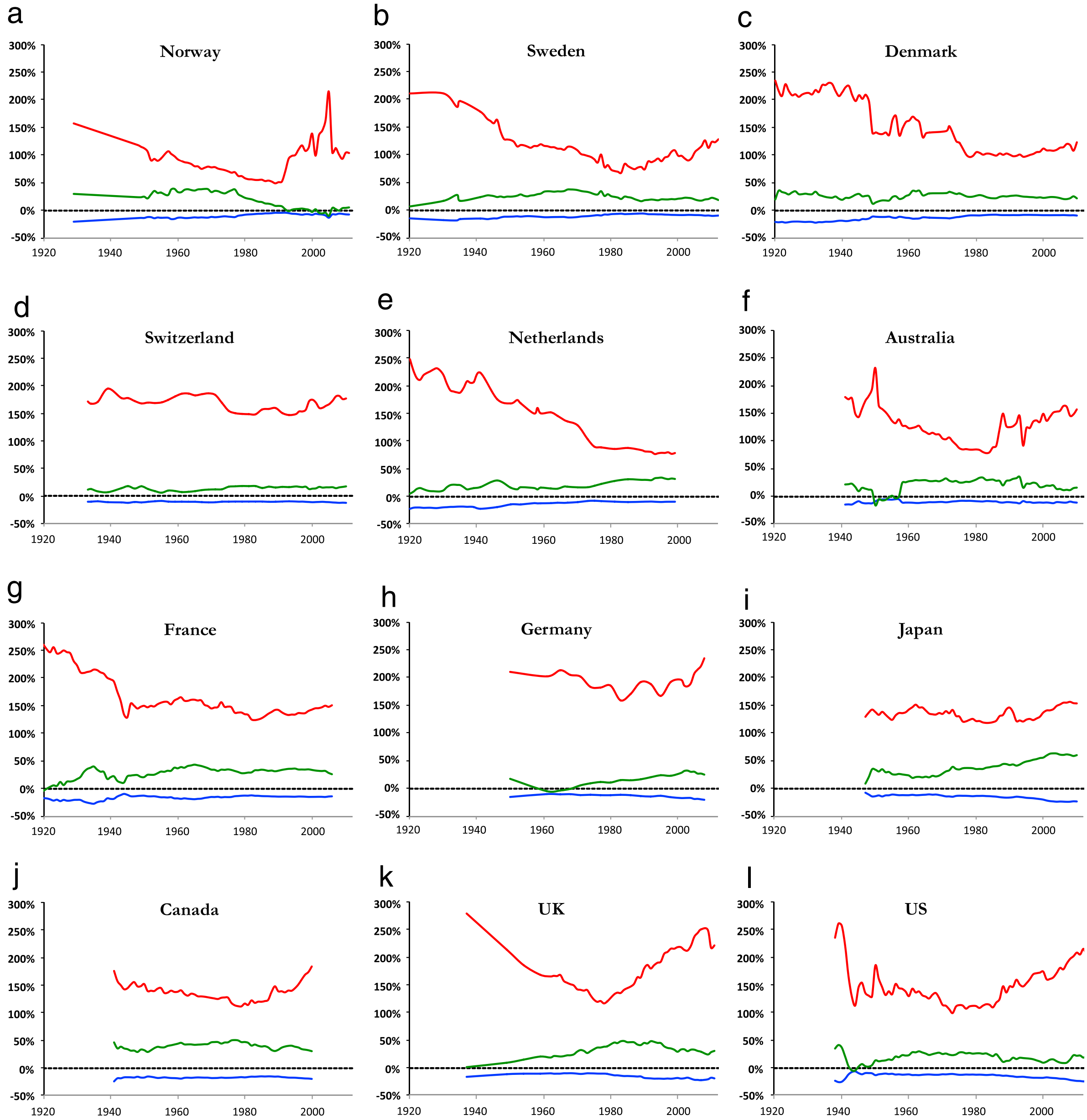


Fig. 4. Global income inequality trends over the years: ψ — Ideal vs Reality. Blue lines: bottom 90%; green lines: top 10%–1%; red lines: top 1%; black dashed lines: 0% ideal reference.

We computed the predicted income shares for the different countries for the period from ∼1920 to ∼2012, depending on the availability of the data in the WTI Database.

We then computed ψ for the three segments (bottom 90%, top 10%–1%, top 1%) for these countries, annually, for the corresponding time periods (sample values are shown in Table 2). These are plotted in Fig. 4. If a country was functioning as an ideal free market system, as defined by our theory, the corresponding ψ for the three segments would all be 0. This is the reference line which is shown as the 0% line (black dotted line) in the plots.

As one can see, the model's predictions are in general agreement with what is known about these countries regarding their inequalities. The twelve countries are shown, roughly, in the order of generally increasing inequality according to our model. Our objective here is not to rank them in a strict order but to show how the different countries have deviated from ideality.

Let us examine what these charts inform us. As expected, none of them are ideal, but there are some pleasant surprises. Consider Norway as an example. Its bottom 90% and top 10%–1% income shares are remarkably close to the ideal values over the last 20 years.

Its bottom 90% ψ has steadily improved, from a low of about −20% in 1929 to about −2% in 1993, and has been ∼5%–10% below the ideal value in the last ∼20 years. Similarly, its top 10%–1% ψ has come down from a high of ∼40% in 1968 to ∼5% in 2011. In fact, during ∼1991–2011, it has been hugging the ideal line quite closely, sometimes a little bit above and sometimes a little below, typically within a narrow ±6% band.

As for the top 1%, ψ has steadily improved towards the ideal value over a period from ∼1929 (at ∼160%) to ∼52% in ∼1990. After spiking up in 2005, it has come down to ∼104% in 2011. But, clearly, top 1%'s share is much more nonideal than that of the bottom 99%.

We find similar close-to-ideality trends in Sweden, Denmark, and Switzerland, for the bottom 90% and top 10%–1%. In these countries, typically, the bottom 90% ψ is within ∼10% of the ideal value; the top 10%–1% ψ is within ∼15%–25%.

All these countries, which practice free market economies, are generally known to be more fair in their economic treatment of all their citizens. *But how fair are they?* We could not answer this question before as there was no reference to compare with, but now we can as our model provides such a benchmark. We find it remarkable that the income sharing these countries have accomplished for their bottom 90% and top 10%–1% are so close to the ideal distributions. We find this to be a surprising result because we did not expect any real-world economic system to come this close to ideality given the simplifying assumptions and approximations in our model. For an overwhelming majority of the population (∼99%), these countries have achieved a near-ideal degree of fairness, presumably through an enlightened combination of individual, corporate and societal values, and macroeconomic policies, all executed through the free market mechanism. Clearly, this did not happen quickly and it took some time, as the trends show, but it is encouraging to find that through mistakes and lessons learnt, societies can evolve and adapt to "discover" a near-ideal distribution, through the free market mechanism, given a chance through the political process. In addition, what is even more remarkable is that these free market economies *did not know*, *a priori*, what the ideal, theoretically fairest, distribution was, and *yet they seem to have "discovered" and maintained a near-ideal outcome empirically on their own*. While these agreements with the aggregate model predictions are encouraging, more thorough studies are needed using detailed distribution data to validate these initial impressions and understand the comparisons better.

From the charts, it appears that Netherlands, Australia and France are broadly in the same general class of higher inequality compared to the first group. Next group is Germany, Japan and Canada, and, finally, UK and US are about the same. It is curious, though, that Japan shows a much higher share for the 10%–1%, compared with even the US or UK. It will be interesting to understand why and how this happens in Japan. Like we noted above, our objective here is not to rank these countries in any strict order, but to show how different countries deviate from ideality.

Another interesting result is that from ∼1945–1975 the US was only ∼12% below the ideal level for the bottom 90%. But, since then, it has lost a lot of ground ending at ∼24% below the ideal level in 2012. It is also interesting that the top 10%–1% dropped from a high of ∼30% in 1963 to a low of 8% in 2007. While these two segments lost ground (strictly speaking, the top 10%–1% is still doing well, enjoying more than its fair share of income), the top 1% went from a low of ∼100% in 1973 to a high of ∼215% in 2012.

Economists have known that the period ∼1945–1975 was when both the bottom and middle classes were doing well, but we see here how well they were doing in enjoying the fruits of the country's progress. While the country might have been more unjust in racial and gender equalities in that period compared to now, it appears to have been closer to ideality in economic matters. Again, further studies are needed to understand why and how we lost that sense of economic fairness, and how to restore it.

We are aware that the 1-class model predictions for the top 1% are not as reliable as the ones for the bottom 90%–95%. That said, we find that the Scandinavian countries, which have managed to approach a near-ideal distribution empirically for the bottom ∼99%, seem to allocate ∼50%–100% more than the ideal share for the top 1% (see Fig. 4) during their best periods of fairness. Even the US was in this range, though near the top end. These trends offer us a valuable insight that ∼50%–100% above the 1-class model's ideal value is perhaps the target to strive for to ensure a near-ideal distribution for the top 1% income share. In that regard, all countries have missed their targets for the top 1%, some more some less, in the last 15 years or so.

A common measure used to quantify inequality is the Gini coefficient, which ranges from 0 (when everyone gets the same income, considered to be the fairest distribution) to 1 (when all income goes to a single individual, considered to be the most unfair outcome). While the Gini index has its uses, we disagree with the assertion that the value 0 (when all incomes are equal) defines the fairest outcome. As we described in Section 2, equality of income is *not* the fairest distribution as different people contribute differently, whether in a corporation or a society. Therefore, while lower Gini coefficient values are generally signaling more fairer outcomes, it does not necessarily imply that they can be used to strictly rank order countries according to their Gini coefficients, as we show next.

Since a lognormal distribution is the fairest outcome, for the sake of comparison, we computed its Gini coefficient, which would make it the ideal value that a country should achieve, for the 12 countries (see Fig. 5). The Gini coefficient for a lognormal distribution is given by:

$$Gini = 2\Phi\left(\frac{\sigma}{\sqrt{2}}\right) - 1 \tag{41}$$

where σ is the lognormal standard deviation and Φ is the cumulative density function for standard normal distribution.

The empirical values are obtained from the Organisation for Economic Co-operation and Development and the Luxembourg Income Studies database and sources [76,77]. They correspond to two different time frames (the exact years are not provided in the sources we used) and they show before and after taxes & transfers. As we have seen in Fig. 4, the general trend of Scandinavian countries closely approaching the ideal distribution is found here as well. Their actual coefficients (post taxes and transfers) are very close to the ideal values, and in some cases they have even over compensated and dropped below them, according to one set of data (e.g., Denmark 0.23 vs 0.23; Norway 0.26 vs 0.30; Sweden 0.24 vs 0.26; Switzerland 0.28 vs 0.31) but not the other set (e.g., Denmark 0.33 vs 0.23; Norway 0.37 vs 0.30; Sweden 0.33 vs 0.26; Switzerland 0.31 vs 0.31). This is even the case for the US (0.36 vs 0.41), which is not equitable as we see in Fig. 4. Given our reservations about the Gini coefficient as a measure of equity and fairness, we do not take these discrepancies too seriously and we show the comparison only for the sake of completeness.

Despite our reservations about the Gini coefficient, there is, however, one valuable lesson to be drawn here. The pre and post taxes & transfers values in the Gini coefficients show how macroeconomic policies can be used to achieve more fairness that approaches ideality in practice. In that regard, it would be valuable to compare the post tax & transfers income distributions for the three segments with the model predictions (like we did in Fig. 4 for the pretax income). Since we now know what the target (i.e., fairest) distributions are, we could *rationally* design and fine tune tax & transfer policies that result in the desired, near-ideal, income distribution for a given society.

We could also develop similar guidelines for deciding executive compensation in corporations. For instance, in November 2013, Swiss voters considered and rejected a referendum [78] that would have capped the CEO pay ratio to 1:12. The number 12 was decided rather arbitrarily; Swiss activists felt that the CEO could not make more in one month than what the lowest employee makes in one year. Using our framework, one can examine this more rationally to develop guidelines which are based on fundamental principles of economic fairness rather than arbitrary limits.

Depending on the aggregate data that is available, one can compute ψ for other segments such as deciles and quintiles. One can then compute an overall composite coefficient Ψ, for example, by calculating

$$\Psi = w_{90}\psi_{90} + w_{10-1}\psi_{10-1} + w_1\psi_1 \tag{42}$$

where w may be equally weighted, population weighted, or income weighted. We have not done this because it is not clear, a priori, what weights would reflect the level of fairness (or unfairness) correctly. Careful studies are needed before any recommendation can be along these lines. Another property of the lognormal distribution that can be used for comparison is, of course, the entropy itself, which is given by

$$\frac{1}{2} + \frac{1}{2}\ln(2\pi\sigma^2) + \mu. \tag{43}$$

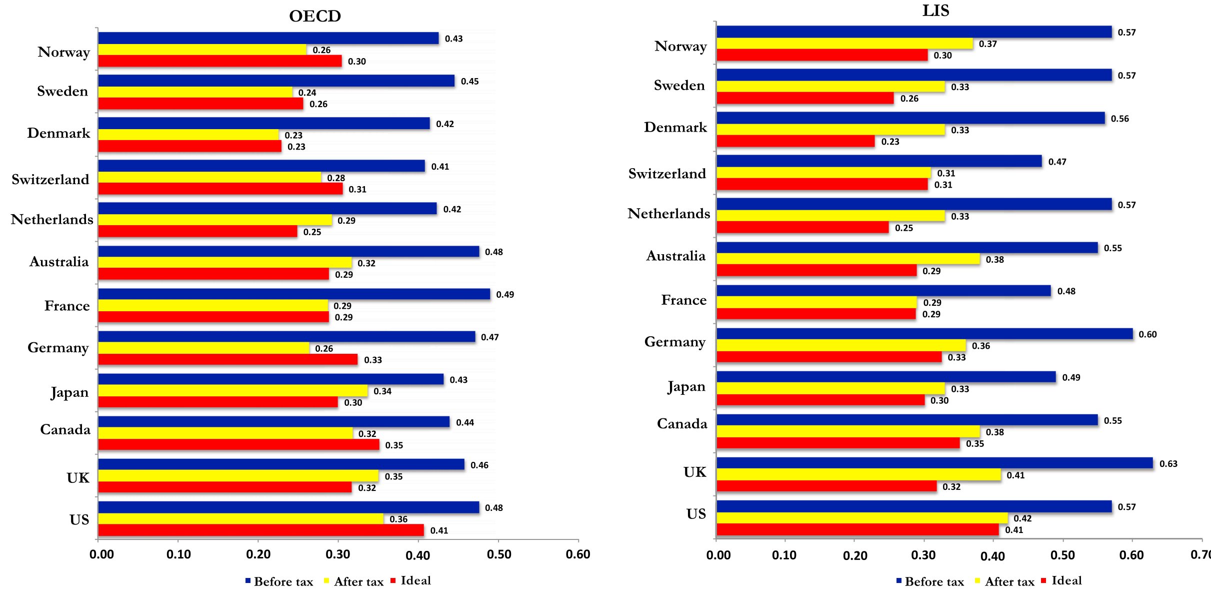
One can then calculate $\psi_{\text{en}} = \textit{Actual entropy}/\textit{Ideal entropy} - 100\%$. We can also develop a similar coefficient by using the Theil Index instead of entropy, which are, of course, closely related. Again, careful further studies are needed to identify the most useful Nonideal Inequality Coefficient. However, it is clear that we need an appropriate reference to compute that, and it is our proposal that we use lognormal distribution as that ideal basis.

4. Discussion and conclusion

There has been some work in the past that explored the connection between game theory and statistical mechanics [79–81]. What is new about our contribution is that it shows a direct and deep connection between the dynamics of animate, fairness-driven, utility-maximizing, rational *teleological* agents and inanimate, purpose-free, thermally-driven molecular entities. Our result reveals the surprising and important connection between entropy and game theoretic potential, demonstrating that the statistical thermodynamic equilibrium reached by molecules is really a Nash equilibrium. We believe that this is a significant insight, for it suggests that statistical thermodynamics can be seen as a special case of potential game theory. Alternatively, one may view this insight as the generalization of the laws of statistical thermodynamics to teleological systems, such as economic systems, yielding a new conceptual framework, which we call *statistical teleodynamics*, that unifies statistical thermodynamics and population game theory. This framework bridges the conceptual gulf mentioned in the introduction, as our ideal teleological agents are rational, fairness-seeking, utility maximizing strategists, with a natural connection to statistical thermodynamics.

As noted, one could presumably choose other expressions to model the three elements in (1), but it is not clear whether they will necessarily lead to the Gibbs–Boltzmann distribution, Helmholtz free energy and entropy in the limiting case of the thermodynamic game involving molecules. We find this correspondence to be particularly appealing, in fact comforting, that statistical teleodynamics properly reduces to well-known results in statistical thermodynamics as a limiting case. This universality has a nice ring to it.

Another important observation is that, in statistical thermodynamics, the claim about the equilibrium state is a *probabilistic* one — it is the *most probable* outcome, one where entropy is maximum. However, our game theoretic result shows that the Nash equilibrium state reached by the molecules, the one that maximizes the potential $\phi(\mathbf{x})$, is a *deterministic* outcome, not a probabilistic one. This observation has potentially important implications concerning the philosophical foundations of statistical thermodynamics, and that of information theory, such as ergodicity and metric transitivity [82,71,83,58,59, 84–86], but we are not addressing them here.



(a) Source: Organisation for Economic Co-operation and Development (OECD) [76].

(b) Source: Luxembourg Income Study (LIS) Database [77].

Fig. 5. Gini coefficients (early to mid 2000s).

As noted in the introduction, researchers in the econophysics community have proposed thermodynamical models for the emergence of income and wealth distributions [30,70,33,32,20,25,23,24]. Even though our contribution also utilizes concepts from statistical mechanics, it takes an entirely different perspective by addressing the fairness issue. The fairness question has not been addressed in the past econophysics approaches. On the other hand, there has been a great amount of work by economists on fairness but these approaches have not addressed whether the free market dynamics will lead to a fair distribution. Indeed, the conventional wisdom in economics is that the free market for labor cares only about efficiency and not fairness. Thus, there is a disconnect between the econophysics and mainstream economics communities in this context. The former has proposed models inspired by statistical mechanical analogues but has not interpreted entropy in economically relevant terms — in particular, it has not addressed the issue of fairness in its theories. In contrast, the latter, which has proposed many theories of fairness, has not recognized the relevance of, and connected with, the statistical thermodynamic theories. Our contribution is to identify the deep connections between these two as well as with game theory, thereby integrating the apparently disparate approaches into a unified conceptual framework.

Econophysicists, typically, [30,23,24] like to claim that the bottom ∼95% follows Boltzmann–Gibbs (BG) exponential distribution or a gamma distribution, not lognormal, and that the top ∼3%–5% follows a Pareto distribution. We beg to differ on both counts as we have shown in this paper. One main difficulty with the BG exponential or the gamma distribution claim is the interpretation of the underlying economic notions. For example, from the maximum entropy procedure which underlie these claims, we can show that the BG exponential distribution implies a utility function that is linear in salary [60] which conflicts with the principle of diminishing marginal utility, one of the founding concepts of economic theory. At the risk of repeating ourselves, we emphasize that in our framework we have tried to formulate an approach that is sensible from a microeconomic perspective — e.g., modeling agents with reasonable utility preferences, rational agents making decisions motivated by utility maximization and not due to random events, recognizing entropy as fairness, etc.

Given that different employees in an organization (or different people in a society) have different talents, thereby making different contributions, some more some less, we expect them to be compensated differently. So, we naturally expect an unequal distribution of pay in an organization. This is only fair as people who contribute more should be paid more. *But how much more? What is the fairest distribution of pay?* In other words, *what is the fairest inequality of pay?* This is at the heart of the inequality debate. We could not answer such questions before. Our theory suggests that the lognormal distribution is the fairest inequality of pay for a homogeneous population. One may view our result as an 'economic law' in the statistical thermodynamics sense. The ideal free market, guided by the "invisible hand", will self-organize to "discover" and obey this economic law if allowed to function freely without collusion or other such unfair interferences subverting the free market dynamics. This result is the economic equivalent of the Gibbs–Boltzmann exponential distribution in thermodynamics.

It is one thing to make such a prediction from theory but another to observe it in practice. So, it is indeed quite remarkable that the Scandinavian free market economies seem to have empirically "discovered" a near-ideal distribution of income for the bottom ∼99% and have operated their economies in that close range for decades. Even the US economy operated a lot closer to ideality, during ∼1945–75, than it does now. It is important to emphasize that in those three decades US performed extremely well economically, dominating the global economy in almost every sector. The lower distribution of income to the top 1% did not disincentivize them from taking calculated risks, innovating, or performing at their peak in their pursuit of income and wealth.

There are obvious limitations to our model — we have assumed perfectly rational agents, no externalities, ideal free market conditions, and so on, which are clearly not valid in real life. However, our objective was to develop a general microeconomic framework, identify key principles, and make predictions that are not restricted by market specific details and nuances. Nevertheless, despite such simplifying assumptions, it is encouraging that our predictions are supported by empirical data. Additional work is needed to examine whether there are other payoff functions which can explain and predict better than what we have proposed. Further research is also needed to determine how to estimate the parameters for the 2-class model so that comparisons can be made with the empirical distributions. The next steps are also to conduct more comprehensive studies of pay distributions in various organizations, and income distributions in different countries, in order to understand in greater detail the deviations from ideality in the market place. Agencies such as the Bureau of Labor Statistics and National Bureau of Economic Research in the US (and similar agencies in other countries, World Bank, etc.) could organize task forces to gather pay data from various companies and organizations. The data should be so grouped to analyze pay distribution patterns across several dimensions such as: (i) organization size — small, medium, large, and very large number of employees, (ii) different industrial sectors, (iii) different types such as private corporations, governments (state and federal), non-profit organizations, etc. Similar studies should be conducted in other countries as well so that we can better understand global patterns.

Further studies are also needed to compare the model predictions with post tax & transfer income data from different countries. Clearly, it is important to understand why and how the deviations from ideality occur in real life. To address this, in addition to gathering empirical data, it would be good to build a large-scale agent-based simulation program (along the lines we presented in 3.4.2) that also accounts for taxes, savings rates, returns on assets etc. Such a program can help us carry out various "what if" scenarios to test the effects of different tax & transfer policies.

As we all know, free markets can sometimes go to the extremes, such as in asset bubbles, or be subverted by unfair practices such as collusion, which lead to unstable and unproductive outcomes for the society at large. Many free market societies have learnt, over the years, that it is in their best interest to strategically and judicially intervene and regulate the market mechanism to avoid such macroscopic instabilities and unproductive consequences. In a similar manner, we believe the current excesses in income inequality and executive compensation are the results of a subverted free market system. As others have argued [1,3–5,8,87], it is imperative to consider strategic interventions to correct the increasingly worrisome income and wealth inequality to improve free market's performance and societal function. By defining and identifying the ideal outcome, we hope that our theory provides an intellectual framework that could be suitably adapted for carefully designing such interventions through macroeconomic policies.

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VV designed the research; VV, YL and JS did the game theoretic analysis; VV and YL performed the data analysis; VV wrote the paper with input from YL and JS.

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